A new picture of the Lifshitz critical behavior

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Abstract

New field theoretic renormalization group methods are developed to describe in a unified fashion the critical exponents of an m-fold Lifshitz point at the two-loop order in the anisotropic ($m \neq d$) and isotropic ($m = d$ close to 8) situations. The general theory is illustrated for the N-vector ϕ^4 model describing a d-dimensional system. A new regularization and renormalization procedure is presented for both types of Lifshitz behavior. The anisotropic cases are formulated with two independent renormalization group transformations. The description of the isotropic behavior requires only one type of renormalization group transformation. We point out the conceptual advantages implicit in this picture and show how this framework is related to other previous renormalization group treatments for the Lifshitz problem. The Feynman diagrams of arbitrary loop-order can be performed analytically provided these integrals are considered to be homogeneous functions of the external momenta scales. The anisotropic universality class (N, d, m) reduces easily to the Ising-like (N, d) when $m = 0$. We show that the isotropic universality class (N, m) when m is close to 8 cannot be obtained from the anisotropic one in the limit $d \rightarrow m$ near 8. The exponents for the uniaxial case $d = 3$, $N = m = 1$ are in good agreement with recent Monte Carlo simulations for the ANNNI model.

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I. INTRODUCTION

Formulated for the first time in 1975 by Hornreich, Luban and Shtrikman in the context of magnetic systems [1,2], the Lifshitz critical behavior has encountered applications in many real physical systems. Some examples include high-T_c superconductors [3, 4, 5], ferroelectric liquid crystals [6, 7, 8], uniaxial ferroelectrics [9], some types of polymers [10, 11, 12, 13] and magnetic materials [14, 15, 16, 17]. In particular, the confluence of a disordered, a uniformly ordered and a modulated ordered phase characterizes the special critical point associated to this critical behavior, known as the Lifshitz point. The modulated phase possesses a fixed equilibrium wavevector which vanishes continuously as the Lifshitz point is approached. When the components of this wavevector span an m-dimensional subspace, the system under consideration displays an m-fold Lifshitz critical behavior. When the order parameter has N components, and the space dimension of the system is d, the Lifshitz universality class is characterized by the set (N, d, m) . Whether $m \neq d$, the system presents the anisotropic Lifshitz critical behavior. Otherwise, the $m = d$ case denotes the isotropic Lifshitz critical behavior. The isotropic case $m = d$ near 8 can be treated, using similar theoretical tools, along the same lines of the anisotropic case. Thus, we shall focus our attention in these two types of critical behavior.

In magnetic systems, the uniaxial ($m = 1$) Lifshitz behavior can be described by an axially next-nearest-neighbor Ising model (ANNNI) [18, 19], which consists of a spin-1/2 Ising model on a cubic lattice with nearest-neighbor interactions as well as next-nearest-neighbor antiferromagnetic couplings along one single lattice axis, the competing axis. The competition between the ferro and antiferromagnetic interactions in this system provokes a different critical behavior when compared to the pure Ising-like behavior. The magnetic compound MnP has been studied extensively in recent years, confirming the appearance of the uniaxial ($m = 1$) Lifshitz behavior which was obtained from theoretical [14, 15, 16] as well as experimental [17] investigations.

This model can be generalized by allowing the next-nearest-neighbor antiferromagnetic couplings along m directions, which represents a typical m-axial Lifshitz critical behavior. In case $m \neq d$, the system naturally admits two independent correlation lengths, namely ξ_{L2} associated to spatial directions perpendicular to the competing axes, and ξ_{L4} associated to directions parallel to the m-dimensional competition subspace. At the Lifshitz point these two correlation lengths become related. In the isotropic behavior $m = d$ close to 8, there is only one correlation length ξ_{L4}.

The field-theoretical representation of this model can be expressed in terms of a modified $\lambda\phi^4$ theory containing higher derivative terms along the m-competing directions. It is given by the following bare Lagrangian density [20] :

$$L = \frac{1}{2}| \bigtriangledown_m^2 \phi_0 |^2 + \frac{1}{2}| \bigtriangledown_{(d-m)} \phi_0 |^2 + \delta_0 \frac{1}{2}| \bigtriangledown_m \phi_0 |^2 + \frac{1}{2} t_0 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4. \tag{1}$$

The field theory treatment turns out to be simpler at the Lifshitz point, where $T = T_L$ and $\delta_0 = 0$. In particular, the functional integral representation permits, at the Lifshitz point, the decoupling of the momentum integrals parallel and perpendicular to the competing axes. It would be interesting to find out whether this condition could make it possible the evaluation of Feynman diagrams to any desired order in a perturbative expansion. Then, the

critical properties of the system, like critical exponents, amplitude ratios and other universal amounts could be obtained analytically utilizing the renormalization group analysis along with ϵ_L-expansion methods. We shall consider this problem from a rather new perspective, which allows a solution in perturbation theory, and which provides an analytic tool that may prove useful in order to figure out the Lifshitz critical behavior in its complete generality.

In this work we present a detailed construction of the new renormalization group description for the anisotropic and isotropic cases. This approach was inspired by an earlier suggestion made by Wilson [21] in order to obtain the critical exponents corresponding to correlations parallel or perpendicular to the competing axes in a manifestly independent manner. This framework was set forth in a previous letter [22]. We discuss the fundamental issues concerning this new renormalization group (RG) analysis leading to new scaling relations to the isotropic case, whereas in the anisotropic case it is shown that these relations are equivalent to previous scaling laws already derived.

In the anisotropic behaviors, the existence of the correlation lengths ξ_{L2} and ξ_{L4} induces two independent characteristic external momenta scales which in turn are used to fix the renormalized field theory in the infrared regime. Since the theory is massless in the Lifshitz critical temperature T_L, the renormalized vertex parts have to be defined at nonvanishing external momenta scales. We denote by κ_1 the external momenta scale along the quadratic (noncompeting) $(d-m)$ directions, whereas κ_2 is the external momenta scale along the quartic (competing) m-dimensional subspace. These external momenta scales originate two independent renormalization group flows in the parameter space. The renormalized coupling constants flow to two independent fixed points, depending whether the renormalization group transformation is over κ_1 or κ_2. At the loop order considered here they are shown to be the same, but it is suggested that this feature is preserved when the analysis is carried out for arbitrary loops. On the other hand, the isotropic case is characterized solely by the correlation length ξ_{L4}. It induces only one characteristic external momenta scale, denoted here by κ_3 which is used to fix the renormalized vertex parts.

Moreover, we calculate all the critical exponents at least at $O(\epsilon_L^2)$ using a novel technique of solving higher-loop Feynman diagrams inspired in this new renormalization group program. Our analysis is performed entirely in momentum space, which is particularly suitable to tackle this problem. The Feynman diagrams are carried out with the help of a new approximation in the quartic momenta subspace, which is the most general approximation consistent with homogeneity. We will show how a former two-loop approximation presented earlier in the calculation of the critical exponents perpendicular to the competing axes [20,23] can be understood in terms of those calculated here using this more elaborate procedure. It is also shown that the relations among the correlation length exponents parallel and perpendicular to the competing axes, namely $\nu_{L4} = \frac{1}{2}\nu_{L2}$, and the anomalous dimensions of the fields, $\eta_{L2} = \frac{1}{2}\eta_{L4}$ are exact at the loop order considered in the present paper. This confirms the strong anisotropic scale invariance predicted before for this sort of system [24].

In Sec. II we set the formalism by defining the normalization conditions for the m-axial Lifshitz critical behavior for the anisotropic and isotropic criticalities. We show that two sets of normalization conditions can naturally describe the anisotropic situation without the need of introducing other dimensionful constants into the analysis.

Section III contains the discussion of the new renormalization analysis for the anisotropic case in directions perpendicular to the competition axes, as well as the new renormalization

group description along directions parallel to the competing subspace.

The renormalization group treatment for the isotropic case is the subject of Sec. IV. The scaling laws are then obtained for this type of critical behavior. Since the scaling relations are different from those appearing in the anisotropic behavior, we point out that the isotropic and anisotropic behaviors are independent and cannot be obtained from each other.

The evaluation of Feynman integrals is presented in Sec. V. We perform the one-, two-, and three-loop integrals using two different approximations. The first approximation introduced in [20, 23] is suitable to perform two- and three-loop integrals in order to obtain the critical exponents perpendicular to the competing axes, since it preserves the homogeneity of the Feynman integrals in the external momenta perpendicular to the competing axes [25]. On the other hand, a new approximation is presented here which preserves the homogeneity of the Feynman diagrams not only in the external momenta perpendicular to the competing axes, *but also in the external momenta parallel to the competing m-dimensional subspace.* Using a simple condition in the competing subspace, we calculated these integrals for arbitrary external momenta.

In Sec. VI we calculate all the critical exponents for the anisotropic case using the scaling relations derived in Sec. III. It would be interesting to obtain the critical exponents using more than one renormalization condition in order to check their correctness. This is done in this section and in the following one. We also discuss our results comparing with alternative field theoretic treatments and new Monte Carlo simulations in $d = 3$ in the context of the ANNNI model ($m = 1$).

Section VII presents the calculation of all the critical exponents for the isotropic case utilizing the new scaling relations obtained in Sec. IV. The analytical expressions obtained are new and to our knowledge are presented here for the first time.

Finally, the conclusions are presented in Sec. VIII and further applications of the method described in this work are pointed out.

II. NORMALIZATION CONDITIONS FOR THE LIFSHITZ CRITICAL BEHAVIOR

From the bare Lagrangian given in (1) we can define renormalized quantities in terms of bare ones through the use of renormalization constants, or renormalization functions. Here we shall follow closely the standard $\lambda\phi^4$ field-theoretic approach. The interested reader should consult, for example, the Amit's book [26] or the original work by Brézin, Le Guillou and Zinn-Justin [27]. These renormalization functions are fixed by the specification of the renormalization scheme used in order to define the renormalized theory. The renormalization functions are defined in terms of the renormalized reduced temperature and order parameter (magnetization in the context of magnetic systems) as $t = Z_{\phi^2}^{-1} t_0$, $M = Z_{\phi}^{\frac{-1}{2}} \phi_0$ and will depend on Feynman integrals. If the theory is renormalized at the critical temperature ($t = 0$), the infrared divergences instruct us to renormalize the theory in nonvanishing external momenta. Therefore, the renormalization constants at the critical temperatute T_L will depend on the external momenta scales involved in the renormalization program.

We first consider the anisotropic behaviors. The Feynman integrals depend on two external momenta scales. We find convenient to define two sets of normalization conditions appropriate to calculate the critical exponents associated to correlations either perpendicular

to or along the competing axes [22]. These external momenta scales were defined above to be κ_1 and κ_2, respectively.

In order to make the calculation easier wherever more than one momentum remains finite, we choose the momenta at a symmetry point (SP). The normalization conditions which yield the critical exponents associated to correlations perpendicular to the competition axes are given by first setting all the external momenta along the competition axes to zero ($\kappa_2 = 0$). Let p_i be the external momenta perpendicular to the competition axes and associated to a generic 1PI vertex part. Then, the external momenta along the quadratic directions are chosen as $p_i.p_j = \frac{\kappa_1^2}{4}(4\delta_{ij} - 1)$. This leads to $(p_i + p_j)^2 = \kappa_1^2$ for $i \neq j$. The momentum scale of the two-point function is fixed through $p^2 = \kappa_1^2 = 1$. Thus, we have the following set of renormalized 1PI vertex parts:

$$\Gamma_R^{(2)}(0, g_1) = 0, \tag{2a}$$

$$\frac{\partial \Gamma_R^{(2)}(p, g_1)}{\partial p^2}|_{p^2=\kappa_1^2} = 1, \tag{2b}$$

$$\Gamma_R^{(4)}(p_i, g_1)|_{SP} = g_1, \tag{2c}$$

$$\Gamma_R^{(2,1)}(p_1, p_2, p, g_1)|_{\bar{SP}} = 1, \tag{2d}$$

$$\Gamma_R^{(0,2)}(p, g_1)|_{p^2=\kappa_1^2} = 0. \tag{2e}$$

Recall that the symmetry point implies that the insertion momentum in Eq.(2d) satisfies $p^2 = (p_1 + p_2)^2 = \kappa_1^2$.

The suitable normalization conditions to dealing with exponents along the competition axes are defined in a similar fashion. Firstly, one sets all the external momenta perpendicular to the competition axes to zero ($\kappa_1 = 0$). If k'_i is the external momenta along the competition axes associated to a generic 1PI vertex part, the external momenta along the quartic directions are chosen as $k'_i.k'_j = \frac{\kappa_2^2}{4}(4\delta_{ij} - 1)$. This implies that $(k'_i + k'_j)^2 = \kappa_2^2$ for $i \neq j$. The momentum scale of the two-point function is fixed through $k'^2 = \kappa_2^2 = 1$. The analogous set of renormalized 1PI vertex parts is given by:

$$\Gamma_R^{(2)}(0, g_2) = 0, \tag{3a}$$

$$\frac{\partial \Gamma_R^{(2)}(k', g_2)}{\partial k'^4}|_{k'^4=\kappa_2^4} = 1, \tag{3b}$$

$$\Gamma_R^{(4)}(k'_i, g_2)|_{SP} = g_2, \tag{3c}$$

$$\Gamma_R^{(2,1)}(k'_1, k'_2, k', g_2)|_{\bar{SP}} = 1, \tag{3d}$$

$$\Gamma_R^{(0,2)}(k', g_2)|_{k'^4=\kappa_2^4} = 0. \tag{3e}$$

Note that, in principle, these two systems of normalization conditions seem to provide two renormalized coupling constants, which arise as a consequence of the two independent flow in the renormalization momenta scales κ_1 and κ_2. Apparently the whole description works with two coupling constants, namely $g_1 = u_1(\kappa_1^2)^{\frac{\epsilon_L}{2}}$ (and $\lambda_1 = u_{01}(\kappa_1^2)^{\frac{\epsilon_L}{2}}$) associated to the flow in the momenta components perpendicular to the m-dimensional axes, as well as $g_2 = u_2(\kappa_2^4)^{\frac{\epsilon_L}{2}}$ (and $\lambda_2 = u_{02}(\kappa_2^4)^{\frac{\epsilon_L}{2}}$) associated to the flow in the momenta components parallel to the m-dimensional axes. Nevertheless, as will be shown, the situation simplifies at the fixed point: both couplings will flow to the same fixed point, at two-loop level,

indicating that this must be so in higher-loop calculations. The conceptual advantage is to treat independently the flow in the momenta along and perpendicular to the competition axes using these two coupling constants. Whether this can be done in a consistent manner is a separate problem, to be tackled in Sec. VI.

The normalization conditions for the isotropic case ($m = d$ near 8) can be defined analogously as those parallel to the competition axes for the anisotropic case. The symmetry point is chosen as follows. If k'_i is the external momenta along the competition axes, the external momenta along the quartic directions are chosen as $k'_i.k'_j = \frac{\kappa_3^2}{4}(4\delta_{ij} - 1)$. This implies that $(k'_i + k'_j)^2 = \kappa_3^2$ for $i \neq j$. The momentum scale of the two-point function is fixed through $k'^4 = \kappa_3^4 = 1$. Then we have the following conditions:

$$\Gamma_R^{(2)}(0, g_3) = 0, \tag{4a}$$
$$\frac{\partial \Gamma_R^{(2)}(k', g_3)}{\partial k'^4}|_{k'^4=\kappa_3^4} = 1, \tag{4b}$$
$$\Gamma_R^{(4)}(k'_i, g_3)|_{SP} = g_3, \tag{4c}$$
$$\Gamma_R^{(2,1)}(k'_1, k'_2, k', g_3)|_{\bar{SP}} = 1, \tag{4d}$$
$$\Gamma_R^{(0,2)}(k', g_3)|_{k'^4=\kappa_3^4} = 0. \tag{4e}$$

Notice that we have not mentioned the quadratic momenta scale κ_1 in the discussion of the isotropic behavior, for it is absent in this situation due to the Lifshitz condition $\delta_0 = 0$.

We can write all the renormalization functions and bare coupling constants in terms of the dimensionless couplings. Let the label $\tau = 1, 2, 3$ refer to the different external momenta scales involved in the general Lifshitz critical behavior, as discussed above for different normalization conditions in the anisotropic and isotropic cases. By expanding the dimensionless bare coupling constants $u_{o\tau}$ and the renormalization functions $Z_{\phi(\tau)}$, $\bar{Z}_{\phi^2(\tau)} = Z_{\phi(\tau)} Z_{\phi^2(\tau)}$ in terms of the dimensionless renormalized couplings u_τ up to two-loop order as

$$u_{o\tau} = u_\tau(1 + a_{1\tau} u_\tau + a_{2\tau} u_\tau^2), \tag{5a}$$
$$Z_{\phi(\tau)} = 1 + b_{2\tau} u_\tau^2 + b_{3\tau} u_\tau^3, \tag{5b}$$
$$\bar{Z}_{\phi^2(\tau)} = 1 + c_{1\tau} u_\tau + c_{2\tau} u_\tau^2, \tag{5c}$$

along with dimensional regularization will be sufficient to determine all the critical exponents.

III. RENORMALIZATION GROUP ANALYSIS FOR THE ANISOTROPIC CASE

Given one bare theory, described by the Lagrangian (1), different versions of the renormalized vertices can be constructed out of the original bare vertex parts. We shall explore now the freedom left in the definition of the renormalization momenta scales κ_1 and κ_2 in the critical theory explained in the last section for the anisotropic case.

We start by considering the renormalization group analysis along directions perpendicular to the competing axes. The renormalized theory is defined with only one quadratic nonvanishing external momenta scale κ_1. Let Λ_1 be the associated cutoff corresponding

to this subspace. The renormalized vertex parts for this case are defined in terms of the normalization constants and the bare vertices as :

$$\Gamma_{R(\tau)}^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, g_\tau, \kappa_\tau) = Z_{\phi(\tau)}^{\frac{N}{2}} Z_{\phi^2(\tau)}^{L} (\Gamma^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, \lambda_\tau, \Lambda_\tau) \\ -\delta_{N,0}\delta_{L,2}\Gamma_{(\tau)}^{(0,2)}(Q_{(\tau)}, Q_{(\tau)}, \lambda_\tau, \Lambda_\tau)|_{Q_{(\tau)}^2=\kappa_\tau^2}) \tag{6}$$

where $p_{i(\tau)}$ $(i = 1, ..., N)$ are the external momenta associated to the vertex functions $\Gamma_{R(\tau)}^{(N,L)}$ with N external legs and $Q_{i(\tau)}$ $(i = 1, ..., L)$ are the external momenta associated to the L insertions of ϕ^2 operators. We emphasize that $p_{i(1)}$ $(i = 1, ..., N)$ refers to the external momenta components along the $(d - m)$ dimensional subspace perpendicular to the competition axes, whereas $p_{i(2)}$ are the external momenta components along the m-dimensional competing subspace. From our normalization conditions, it should be kept in mind that all quantities presenting a subscript $\tau = 1(2)$ are calculated at zero external momenta components parallel (perpendicular) to the competing axes and are characterized by the momenta scale $\kappa_1(\kappa_2)$. From the last section, $u_{0\tau}$, $Z_{\phi(\tau)}$ and $Z_{\phi^2(\tau)}$ are represented as power series in u_τ. The renormalization group invariance of the bare vertex with the momenta scale κ_τ implies that :

$$(\kappa_\tau \frac{\partial}{\partial \kappa_\tau})_{\lambda_\tau,\Lambda_\tau}[Z_{\phi(\tau)}^{-\frac{N}{2}} Z_{\phi^2(\tau)}^{-L}(\Gamma_{R(\tau)}^{(N,L)} - \delta_{N,0}\delta_{L,2}\Gamma_{(\tau)}^{(N,L)})] = 0. \tag{7}$$

This in turn yields the following RG equations:

$$(\kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \bar{\beta}_\tau \frac{\partial}{\partial g_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(g_\tau, \kappa_\tau) + \\ L\gamma_{\phi^2(\tau)}(g_\tau, \kappa_\tau))\Gamma_{R(\tau)}^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, \lambda_\tau, \Lambda_\tau) = \delta_{N,0}\delta_{L,2}(\kappa_\tau^{-2\tau})^{\frac{\epsilon_L}{2}} B_\tau, \tag{8}$$

where B_τ is a constant used to renormalize $\Gamma_{R(\tau)}^{(0,2)}$ and

$$\bar{\beta}_\tau(g_\tau, \kappa_\tau) = (\kappa_\tau \frac{\partial g_\tau}{\partial \kappa_\tau})_{\lambda_\tau,\Lambda_\tau} \tag{9a}$$

$$\gamma_{\phi(\tau)}(g_\tau, \kappa_\tau) = (\kappa_\tau \frac{\partial ln Z_{\phi(\tau)}}{\partial \kappa_\tau})_{\lambda_\tau,\Lambda_\tau} \tag{9b}$$

$$\gamma_{\phi^2(\tau)}(g_\tau, \kappa_\tau) = (\kappa_\tau \frac{\partial ln Z_{\phi^2(\tau)}}{\partial \kappa_\tau})_{\lambda_\tau,\Lambda_\tau}. \tag{9c}$$

are functions of g_τ and κ_τ only, though they are functions of Λ_τ implicitly. Notice that $\Gamma_R^{(0,2)}$ is different from all other vertices since the RG equation presents an inhomogeneous term in the left hand side due to its additive renormalization. We shall treat this additively renormalized vertex part later on. The above expressions correspond to the limit $\Lambda_\tau \rightarrow \infty$, which are naturally finite, even if $\lambda_\tau, Z_{\phi(\tau)}$ and $Z_{\phi^2(\tau)}$ diverge at this limit. It is worth expressing all of these quantities in terms of dimensionless bare and renormalized coupling constants. We now turn our attention to discuss the central issue of the new dimensional considerations which will be useful for the subsequent dimensional analysis.

Consider the volume element in momentum space for calculating an arbitrary Feynman integral. It is given by $d^{d-m}qd^mk$, where $\vec{q}$ represents a $(d - m)$-dimensional vector

perpendicular to the competing axes and $\vec{k}$ denotes an m-dimensional vector along the competing subspace, respectively. The Lifshitz condition $\delta_0 = 0$ suppresses the quadratic part of the momentum along the competition axes. Nevertheless, there is still a contribution from the quartic momenta contained in the inverse critical ($t = 0$) free propagator $G_0^{(2)-1}(q, k) = (k^2)^2 + q^2$. In order to be dimensionally consistent, the canonical dimension in mass units of both terms in the propagator should be equal. There are two ways out of this outstanding situation.

The former idea, inspired in Ref. [1], is to introduce a dimensionful constant σ in front of the first term in the Lagrangian (1), along with its renormalization function Z_σ, as was done by Mergulhão and Carneiro [28]. This idea implies that the momenta scales parallel or perpendicular to the competition directions play the same role in this discussion and there is only one coupling constant. Denoting the components of the quartic external momenta with a subscript α and the quadratic components with a subscript β, they set the following renormalization conditions:

$$
\begin{aligned}
&\frac{\partial \Gamma_R^{(2)}(k, -k, \sigma, g, \kappa)}{\partial k_\beta^2}|_{k_\beta^2 = \kappa^2} = 1 \\
&\frac{\partial \Gamma_R^{(2)}(k, -k, \sigma, g, \kappa)}{\partial k_\alpha^4}|_{\sigma k_\alpha^4 = \kappa^2} = \sigma, \\
&\Gamma_R^{(4)}(k_i, \sigma, g, \kappa)|_{SP_\alpha} = g, \\
&\Gamma_R^{(2,1)}(k_1, k_2, p, \sigma, g, \kappa)|_{S\bar{P}_\alpha} = 1, \\
&\Gamma_R^{(0,2)}(p, -p, \sigma, g, \kappa)|_{\sigma p_\alpha^4 = \kappa^2} = 0,
\end{aligned}
$$

where the SP_α means $\sigma^{\frac{1}{2}} k_{i\alpha} k_{j\alpha} = \kappa \frac{(4\delta_{ij} - 1)}{4}$ and was chosen at zero quadratic external momenta. This choice of renormalization points makes the renormalization constants σ independent as claimed by those authors [29]. However, σ is still a relevant length (momentum) scale and this fact should be reflected on its dependence in some normalization constants. Therefore, starting with a dimensionful σ parameter and making it dimensionless in the end of the calculation as they chose does not seem to be consistent, since the quartic and quadratic momenta scale play the same role and have the same canonical dimensions in this approach. Notice that the last 4 of these equations together with the critical theory condition on the renormalized two-point vertex part naturally defines an independent set of normalization conditions along the competing axes. In fact, if the quartic momenta is redefined through $\sigma k_\alpha^4 \equiv k_\alpha'^4$ such that σ is absorbed in the new quartic momenta, this implies that $\sigma^{\frac{1}{2}} k_{i\alpha} k_{j\alpha} = k'_{i\alpha} k'_{j\alpha} = \kappa'^2 \frac{(4\delta_{ij} - 1)}{4}$ with $\kappa' \neq \kappa$. Then, one has 5 normalization conditions along the quartic subspace as described in the last section.

On the other hand, the first of these equations is calculated at $\sigma k_\alpha^4 = 0$. Intuitively it should be complemented with 4 more normalization conditions with nonvanishing external quadratic momenta perpendicular to the competing subspace. This is what was done in the last section for directions perpendicular to the competing axes. Thus, if we have two different momenta scales κ and κ' and setting them equal is equivalent to have Mergulhão and Carneiro's renormalization conditions, with 5 more normalization conditions along the quadratic directions. Thus if one trades σ by an independent external quartic momenta scale κ', it still remains the 5 extra normalization conditions which in their approach are

undefined. Nevertheless, they recovered the former anisotropic scaling relations [1] using this reasoning. They used their symmetry point in order to treat the cases $m = 2, 6$ in the context of an ϵ_L-expansion [29].

There is an alternative based in a recently proposed method which does not use the dimensional constant σ but allows the realization of a dimensional redefinition of the momenta components along the quartic competing subspace [22]. This later view inspires the subsequent discussion and shall be used throughout this paper. Let $[\vec{q}] = M$ be the mass dimension of the quadratic momenta. The consistency of the Lagrangian density (1) on dimensional grounds requires that $[\vec{k}] = M^{\frac{1}{2}}$. This is equivalent to perform a dimensional redefinition of the momenta along the competing axes, as long as the condition $\delta_0 = 0$ is satisfied. The volume element in momentum space $d^{d-m}qd^mk$ has mass dimension $[d^{d-m}qd^mk] = M^{d-\frac{m}{2}}$. The dimension of the field is obtained by requiring that the volume integral of the Lagrangian density (1) is dimensionless in mass units. It follows that $[\phi] = M^{\frac{1}{2}(d-\frac{m}{2})-1}$.

The N-point Green function can be expressed dimensionally as $[G^{(N)}(x_1, ..., x_N)] = [\phi]^N = M^{\frac{N}{2}(d-\frac{m}{2})-N}$. The associated one particle irreducible (1PI) vertex parts have dimension in mass units $[\Gamma^{(N)}(x_i)] = [G^{(N)}(x_i)][V]^{-N}[G^{(2)}(x_i)]^{-N} = M^{\frac{N}{2}(d-\frac{m}{2})+N}$. In momentum space, the Fourier transform is obtained by integrating over each one of the coordinates. Removing the momentum conserving δ-function, we have $[\Gamma^{(N)}(k_i)] = M^{N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2}}$.

As usual, the exponent of M in the above relations is called the canonical dimension of the quantity. If the physical quantity O has canonical dimension $[O] = M^{\Delta}$, then under a transformation of the length scale associated to $M \rightarrow \alpha M$ it implies that $O = \alpha^{\Delta} O$. Therefore, all dimensionfull parameters are transformed under a transformation in the lengths (or external momenta). Hence, it is useful to describe the theory in terms of dimensionless parameters. As the coupling constants are associated to $\Gamma^{(4)}$, we can write $g_\tau = u_\tau(\kappa_\tau^{2\tau})^{\frac{\epsilon_L}{2}}$, and $\lambda_\tau = u_{0\tau}(\kappa_\tau^{2\tau})^{\frac{\epsilon_L}{2}}$, where $\epsilon_L = 4 + \frac{m}{2} - d$.

In terms of the dimensionless couplings defined above, the renormalization group equation can be rewritten as:

$$(\kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \beta_\tau \frac{\partial}{\partial u_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(u_\tau) + L \gamma_{\phi^2(\tau)}(u_\tau)) \Gamma_{R(\tau)}^{(N,L)} = \delta_{N,0} \delta_{L,2} (\kappa_\tau^{-2\tau})^{\frac{\epsilon_L}{2}} B_\tau(u_\tau), \tag{11}$$

and from now on we can forget about the cutoffs Λ_τ, bearing in mind, however, that they should be kept fixed in all stages of the analysis. The functions

$$\beta_\tau = (\kappa_\tau \frac{\partial u_\tau}{\partial \kappa_\tau}), \tag{12a}$$

$$\gamma_{\phi(\tau)}(u_\tau) = \beta_\tau \frac{\partial ln Z_{\phi(\tau)}}{\partial u_\tau} \tag{12b}$$

$$\gamma_{\phi^2(\tau)}(u_\tau) = -\beta_\tau \frac{\partial ln Z_{\phi^2(\tau)}}{\partial u_\tau} \tag{12c}$$

are calculated at fixed bare coupling λ_τ. The β_τ-functions can be cast in a more useful form in terms of dimensionless quantities, namely,

$$\beta_\tau = -\tau \epsilon_L (\frac{\partial ln u_{0\tau}}{\partial u_\tau})^{-1}. \tag{13}$$

Note that the beta function corresponding to the flow in κ_2 has a factor of 2 compared to that associated to the flow in κ_1. As usual, they are power series in u_τ, with coefficients which depend on ϵ_L. Let us analyse the simplest case $L = 0$. The solution can be expressed in terms of characteristics. The characteristic equation is given by two independent flows in the coupling constants induced by the flows in the momenta scale κ_1 and κ_2, i.e,

$$\rho_\tau \frac{du_\tau(\rho_\tau)}{d\rho_\tau} = \beta(u_\tau(\rho_\tau)), \tag{14}$$

with the initial condition $u_\tau(\rho_\tau = 1) = u_\tau$. Using the characteristic equation for u_τ we can change variables from a variable x_τ to u_τ, through the relation:

$$\int_1^{\rho_\tau} f(u_\tau(x_\tau)) \frac{dx_\tau}{x_\tau} = \int_{u_\tau}^{u_\tau(\rho_\tau)} \frac{f(u_\tau)}{\beta_\tau(u_\tau)}. \tag{15}$$

Thus, small values of x_τ in the left-hand-side correspond to the neighborhood of the zero's of β_τ in the right-hand-side. For the anisotropic case, the solution to the renormalization group equation reflects the two-parameters group of invariance, and can be expressed in the form

$$\Gamma^{(N)}_{R(\tau)}(k_{i(\tau)}, u_\tau, \kappa_\tau) = exp[-\frac{N}{2} \int_1^{\rho_\tau} \gamma_{\phi(\tau)}(u_\tau(\rho_\tau)) \frac{dx_\tau}{x_\tau}]\, \Gamma^{(N)}_{R(\tau)}(k_{i(\tau)}, u_\tau(\rho_\tau), \kappa_\tau \rho_\tau). \tag{16}$$

From our dimensional analysis, the dimensional redefinition of the momenta along the competing axes results in an effective space dimension for the anisotropic case, i.e., $(d - \frac{m}{2})$. Thus, we find the following behavior for the 1PI vertex parts $\Gamma^{(N)}_{R(\tau)}$ under flows in the external momenta:

$$\Gamma^{(N)}_{R(\tau)}(\rho_\tau k_{i(\tau)}, u_\tau, \kappa_\tau) = \rho_\tau^{\tau(N + (d - \frac{m}{2}) - \frac{N(d - \frac{m}{2})}{2})} exp[-\frac{N}{2} \int_1^{\rho_\tau} \gamma_{\phi(\tau)}(u_\tau(\rho_\tau)) \frac{dx_\tau}{x_\tau}] \\ \Gamma^{(N)}_{R(\tau)}(k_{i(\tau)}, u_\tau(\rho_\tau), \kappa_\tau \rho_\tau). \tag{17}$$

It is helpful to present the explicit expressions for vertex parts calculated either at zero quartic external momenta or at vanishing quadratic external momenta. The renormalized vertex parts calculated at zero quartic external momenta is given by

$$\Gamma^{(N)}_{R(1)}(\rho_1 k_{i(1)}, u_1, \kappa_1) = \rho_1^{N + (d - \frac{m}{2}) - \frac{N(d - \frac{m}{2})}{2}} exp[-\frac{N}{2} \int_1^{\rho_1} \gamma_{\phi(1)}(u_1(\rho_1)) \frac{dx_1}{x_1}] \\ \Gamma^{(N)}_{R(1)}(k_{i(1)}, u_1(\rho_1), \kappa_1 \rho_1). \tag{18}$$

The dependence of the renormalized vertex parts is quadratic in the external momenta perpendicular to the competing axes. Therefore, the analysis is completely similar to the pure $\lambda\phi^4$-theory, with the replacement $\epsilon_L \rightarrow \epsilon$. From this analysis, we can identify the label $\tau = 1$ with the subscript $L2$. Then, we could have written $\gamma_{\phi(1)} \equiv \gamma_{\phi(L2)}$ and $\gamma_{\phi^2(1)} \equiv \gamma_{\phi^2(L2)}$.

On the other hand, the renormalized vertex parts at zero quadratic external momenta can be expressed as

$$\Gamma^{(N)}_{R(2)}(\rho_2 k_{i(2)}, u_2, \kappa_2) = \rho_2^{2(N + (d - \frac{m}{2}) - \frac{N(d - \frac{m}{2})}{2})} exp[-\frac{N}{2} \int_1^{\rho_2} \gamma_{\phi(2)}(u_2(\rho_2)) \frac{dx_2}{x_2}] \\ \Gamma^{(N)}_{R(2)}(k_{i(2)}, u_2(\rho_2), \kappa_2 \rho_2). \tag{19}$$

The difference is that the canonical dimension is twice as big as the canonical dimension of the vertex parts calculated at zero quartic momenta. Then, we can make the identifications $\gamma_{\phi(2)} \equiv \gamma_{\phi(L4)}$ and $\gamma_{\phi^2(2)} \equiv \gamma_{\phi^2(L4)}$. The last equations imply that a change in the external momenta scale is equivalent to the multiplication of the vertex function by that scale to the power of the canonical dimension of the function, followed by a modified coupling constant, which flows with the characteristic equation, and an additional factor.

It is interesting to analyse the vertex functions at the infrared fixed points, since this will determine the scaling laws and the critical exponents associated to correlations perpendicular and parallel to the m-dimensional competing subspace. The analysis can be carried out by assuming that there are two independent fixed points, defined by $\beta_\tau(u_\tau^*) = 0$. The renormalization group equation leads to a simple scaling property at the fixed points. It implies the following solution to the vertex functions:

$$\Gamma_{R(\tau)}^{(N)}(\rho_\tau k_{i(\tau)}, u_\tau^*, \kappa_\tau) = \rho_\tau^{\tau(N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2})-\frac{N}{2}\gamma_{\phi(\tau)}(u_\tau^*)} \Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, u_\tau^*, \kappa_\tau). \tag{20}$$

For $N = 2$, we have

$$\Gamma_{R(\tau)}^{(2)}(\rho_\tau k_{(\tau)}, u_\tau^*, \kappa_\tau) = \rho_\tau^{2\tau-\gamma_{\phi(\tau)}(u_\tau^*)}\Gamma_{R(\tau)}^{(2)}(k_{(\tau)}, u_\tau^*, \kappa_\tau). \tag{21}$$

The quantity $\gamma_{\phi(\tau)}(u_\tau^*)$ can be interpreted in the following way. If the field theory is free, a change in the external momenta scale will produce a change in the free vertex $\Gamma_{(\tau)}^{(N)0}$ which scales with the canonical dimension of the vertex, that is

$$\Gamma_{(\tau)}^{(N)0}(\rho_\tau k_{i(\tau)}) = \rho_\tau^{\tau(N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2})}\Gamma_{(\tau)}^{(N)}(k_{i(\tau)}). \tag{22}$$

We then define the dimension of the field ϕ as

$$\Gamma_{(1)}^{(N)}(\rho_\tau k_{i(\tau)}) = \rho_\tau^{\tau[(d-\frac{m}{2})-Nd_{\phi(\tau)}]}\Gamma_{(1)}^{(N)}(k_i), \tag{23}$$

such that in the free theory $d_\phi^0 = \frac{d-\frac{m}{2}}{2} - 1$ is the naive dimension of the field. At the fixed point, the naive dimension is modified due to the presence of interactions, such that the nontrivial effect is the appearance of the anomalous dimension, i.e., $d_{\phi(\tau)} = \frac{d-\frac{m}{2}}{2} - 1 + \frac{\eta_\tau}{2\tau}$. When $N = 2$, this identifies the anomalous dimensions of the field in the anisotropic situation, namely that associated to the change in the external momenta scale perpendicular to the competing axes $\eta_1 \equiv \eta_{L2} = \gamma_{\phi(1)}(u_1^*)$ as well as the other corresponding to the change in the external momenta parallel to the competition subspace $\eta_2 \equiv \eta_{L4} = \gamma_{\phi(2)}(u_2^*)$.

This can be easily generalized to include L insertions of ϕ^2 operators in quite an analogous way, such that the RG equations at the fixed point lead to the solution $((N, L) \neq (0, 2))$:

$$\Gamma_{R(\tau)}^{(N,L)}(\rho k_{i(\tau)}, \rho p_{i(\tau)}, u_\tau^*, \kappa_\tau) = \rho_\tau^{\tau[N+(d-\frac{m}{2})-\frac{N(d-\frac{m}{2})}{2}-2L]-\frac{N\gamma_{\phi(\tau)}^*}{2}+L\gamma_{\phi^2(\tau)}^*} \times \Gamma_{R(\tau)}^{(N,L)}(k_{i(\tau)}, p_{i(\tau)}, u_\tau^*, \kappa_\tau). \tag{24}$$

Thus, if we write at the fixed point

$$\Gamma_{R(\tau)}^{(N,L)}(\rho k_{i(\tau)}, \rho p_{i(\tau)}, u_\tau^*, \kappa_\tau) = \rho_\tau^{\tau[(d-\frac{m}{2})-Nd_\phi]+Ld_{\phi^2}} \Gamma_{R(\tau)}^{(N,L)}(k_{i(\tau)}, p_{i(\tau)}, u_\tau^*, \kappa_\tau), \tag{25}$$

the anomalous dimensions of the insertions of ϕ^2 operators are given by $d_{\phi^2} = -2\tau + \gamma_{\phi^2(\tau)}(u_\tau^*)$ and will be related to the critical exponents $\nu_1 \equiv \nu_{L2}$ and $\nu_2 \equiv \nu_{L4}$ as we shall see in a moment.

In order to find the scaling relations we must go away from the Lifshitz critical temperature ($t \neq 0$) staying, however, at the critical region $\delta_0 = 0$ [35]. Above the Lifshitz critical temperature, the renormalized vertex parts for $t \neq 0$ can be expanded as a power series in t around those renormalized vertices at $t = 0$, provided $N \neq 0$. We can now apply the differential operators

$$O_\tau = \kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \beta_\tau \frac{\partial}{\partial u_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(u_\tau) + \gamma_{\phi^2(\tau)}(u_\tau) t \frac{\partial}{\partial t} \tag{26}$$

to $\Gamma_{R(\tau)}^{(N)}(k_{i(\tau)})$ such that we find

$$O_\tau \Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, t, u_\tau^*, \kappa_\tau) = \sum_{L=0}^{\infty} \frac{t^L}{L!} [\kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \beta_\tau \frac{\partial}{\partial u_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(u_\tau) + L \gamma_{\phi^2(\tau)}] \tag{27}$$
$$\times \; \Gamma_{R(\tau)}^{(N,L)}(k_{i(\tau)}, p_{i(\tau)}, u_\tau^*, \kappa_\tau),$$

The result is that each term in the sum vanishes because of the RGE for $\Gamma_{R(\tau)}^{(N,L)}(k_{i(\tau)}, p_{i(\tau)}, u_\tau^*, \kappa_\tau)$. Hence, we obtain

$$[\kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \beta_\tau \frac{\partial}{\partial u_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(u_\tau) + \gamma_{\phi^2(\tau)}(u_\tau) t \frac{\partial}{\partial t}] \Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, t, u_\tau^*, \kappa_\tau) = 0. \tag{28}$$

The solution is a homogeneous function of the product of $k_{i(\tau)}$ (to some power) and t solely at the fixed point u_τ^*. As the value of u_τ is fixed at u_τ^*, we shall omit it from the notation of this section from now on. It is given by:

$$\Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, t, \kappa_\tau) = \kappa_\tau^{\frac{N\gamma_{\phi(\tau)}^*}{2}} F_{(\tau)}^{(N)}(k_{i(\tau)}, \kappa_\tau t^{\frac{-1}{\gamma_{\phi^2(\tau)}^*}}). \tag{29}$$

If we define $\theta_\tau = -\gamma_{\phi^2(\tau)}^*$, and using dimensional analysis, we find

$$\Gamma_{R(\tau)}^{(N)}(k_{i(\tau)}, t, \kappa_\tau) = \rho_\tau^{\tau[N+(d-\frac{m}{2})-\frac{N}{2}(d-\frac{m}{2})]-\frac{N}{2}\eta_\tau} \kappa_\tau^{\frac{N}{2}\eta_\tau}$$
$$F_{(\tau)}^{(N)}(\rho_\tau^{-1} k_{i(1)}, (\rho^{-1}\kappa_\tau)(\rho^{-2\tau} t)^{\frac{-1}{\theta_\tau}}). \tag{30}$$

By choosing $\rho_\tau = \kappa_\tau (\frac{t}{\kappa_\tau^{2\tau}})^{\frac{1}{\theta_\tau + 2\tau}}$, and replacing back in (29), the vertex function depends only on the combination $k_{i(\tau)} \xi_\tau$ apart from a power of t. Since the correlation lengths ξ_τ are proportional to $t^{-\nu_\tau}$, we can identify the critical exponents ν_τ as

$$\nu_\tau^{-1} = 2\tau + \theta_\tau^* = 2\tau - \gamma_{\phi^2(\tau)}^*. \tag{31}$$

According to our conventions, these equations are equivalent to the following scaling relations

$$\nu_{L2}^{-1} = 2 - \gamma^*_{\phi^2(1)}, \tag{32a}$$
$$\nu_{L4}^{-1} = 4 - \gamma^*_{\phi^2(2)}. \tag{32b}$$

As a matter of convenience, we could have defined alternatively the function

$$\bar{\gamma}_{\phi^2(\tau)}(u_\tau) = -\beta_\tau \frac{\partial ln(Z_{\phi^2(\tau)} Z_{\phi(\tau)})}{\partial u_\tau}. \tag{33}$$

In that case we would have found the equivalent formulae

$$\nu_{L2}^{-1} = 2 - \eta_{L2} - \bar{\gamma}_{\phi^2(1)}(u_1^*), \tag{34a}$$
$$\nu_{L4}^{-1} = 4 - \eta_{L4} - \bar{\gamma}^*_{\phi^2(2)}. \tag{34b}$$

Hence, at the fixed point all correlation functions (not including composite operators) scale at $T > T_L$, since they are functions of $k_{i(\tau)}\xi_\tau$ only. For $N = 2$ we choose $\rho_\tau = k_{(\tau)}$, the external momenta. Then $\Gamma^{(2)}_{R(\tau)}(k_{(\tau)}, t, \kappa_\tau) = k^{2\tau-\eta_\tau}\kappa_\tau^{\eta_\tau} f(k_{(\tau)}\xi_\tau)$. The critical situation is characterized when $\xi_\tau \to \infty$ and $k_{(\tau)} \to 0$ such that $f(k_{(\tau)}\xi_\tau) \to Constant$. Defining $f_\tau = (k_{(\tau)}\xi_\tau)^{2\tau-\eta_\tau} f(k_{(\tau)}\xi_\tau)$, we have

$$\Gamma^{(2)}_{R(\tau)}(k_{(\tau)}, t, \kappa_\tau) = (k_{(\tau)}\xi_\tau)^{2\tau-\eta_\tau}\kappa_\tau^{\eta_\tau} f_\tau(k_{(\tau)}\xi_\tau). \tag{35}$$

The susceptibility is proportional to $t^{-\gamma_\tau}$ as $k_{(\tau)} \to 0$. Thus, since $\Gamma^{(2)}_{R(\tau)} = \chi^{-1}_{(\tau)}$, we can identify the susceptibility critical exponents

$$\gamma_\tau = \nu_\tau(2\tau - \eta_\tau). \tag{36}$$

These relations are equivalent to the relations:

$$\gamma_{L2} = \nu_{L2}(2 - \eta_{L2}), \tag{37}$$

$$\gamma_{L4} = \nu_{L4}(4 - \eta_{L4}). \tag{38}$$

The specific heat exponents can be obtained by analysing the RG equation for $\Gamma^{(0,2)}_{R(\tau)}$ above T_L at the fixed point, i.e.

$$(\kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \gamma^*_{\phi^2(\tau)}(2 + t\frac{\partial}{\partial t}))\Gamma^{(0,2)}_{R(\tau)} = (\kappa_\tau^{-2\tau})^{\frac{\epsilon_L}{2}} B_\tau(u_\tau^*), \tag{39}$$

where $B_\tau(u_\tau^*)$ is given by

$$(\kappa_\tau^{-2\tau})^{\frac{\epsilon_L}{2}} B_\tau(u_\tau^*) = -Z^2_{\phi^2(\tau)}\kappa_\tau \frac{\partial}{\partial \kappa_\tau}\Gamma^{(0,2)}_{(\tau)}(Q_{(\tau)}; -Q_{(\tau)}, \lambda_\tau)|_{Q^2_{(\tau)}=\kappa_\tau^2}. \tag{40}$$

It is a inhomogeneous part which has no dependence in the reduced temperature t. The bare vertex function $\Gamma^{(0,2)}_{(\tau)}$ is calculated as before in the limit $\Lambda_\tau \to \infty$, with a fixed bare coupling constant, which renders $B_\tau(u_\tau^*)$ finite in this limit when $(d - \frac{m}{2}) = 4$. This renormalized vertex part consists of the addition of the homogeneous (temperature dependent) and inhomogeneous pieces. The general discussion given so far for the vertex part $\Gamma^{(N,L)}_{R(\tau)}$

will be useful to determine the homogeneous part of the solution. Indeed, at the fixed point the obvious generalization of the solution for $\Gamma_{R(\tau)}^{(N,L)}$ is given by:

$$\Gamma_{R(\tau)}^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, t, \kappa_\tau) = \kappa_\tau^{\frac{1}{2}N\gamma_{\phi(\tau)}^{*} - L\gamma_{\phi^2(\tau)}^{*}} F_\tau^{(N,L)}(p_{i(\tau)}, Q_{i(\tau)}, \kappa_\tau t^{\frac{-1}{\gamma_{\phi^2(\tau)}^{*}}}). \tag{41}$$

Therefore, the temperature dependent homogeneous part for $\Gamma_{R(\tau),h}^{(0,2)}$ will scale at the fixed point as:

$$\Gamma_{R(\tau),h}^{(0,2)}(Q_{(\tau)}, -Q_{(\tau)}, t, \kappa_\tau) = \kappa_\tau^{-2\gamma_{\phi^2(\tau)}^{*}} F_\tau^{(0,2)}(Q_{(\tau)}, -Q_{(\tau)}, \kappa_\tau t^{\frac{-1}{\gamma_{\phi^2(\tau)}^{*}}}). \tag{42}$$

This will be identified with the specific heat at zero external momentum insertion $Q_{(\tau)} = 0$. Using the results of our dimensional analysis

$$\Gamma_{R(\tau),h}^{(0,2)})(Q_{(\tau)}, -Q_{(\tau)}, t, \kappa_\tau) = \rho_\tau^{\tau[(d-\frac{m}{2})-4]+2\gamma_{\phi^2(\tau)}^{*}} \Gamma_{R(\tau),h}^{(0,2)}(\rho_\tau^{-1}Q_{(\tau)}, -\rho_\tau^{-1}Q_{(\tau)}, \rho_\tau^{-2\tau}t, \rho_\tau^{-1}\kappa_\tau), \tag{43}$$

and replacing this into the solution at the fixed point, we find

$$\begin{aligned}\Gamma_{R(\tau),h}^{(0,2)}(Q_{(\tau)}, -Q_{(\tau)}, t, \kappa_\tau) = {}& \rho_\tau^{\tau[(d-\frac{m}{2})-4]+2\gamma_{\phi^2(\tau)}^{*}} \kappa_\tau^{-2\gamma_{\phi^2(\tau)}^{*}} \\ \times {}& F_\tau^{(0,2)}(\rho_\tau^{-1}Q_{(\tau)}, -\rho_\tau^{-1}Q_{(\tau)}, \rho_\tau^{-1}\kappa_\tau(\rho_\tau^{-2\tau}t)^{\frac{-1}{\gamma_{\phi^2(\tau)}^{*}}}).\end{aligned} \tag{44}$$

Again we choose $\rho_\tau = \kappa_\tau(\frac{t}{\kappa_\tau^{2\tau}})^{\frac{1}{\theta_\tau+2\tau}}$. Substituting this in last equation, taking the limit $Q_{(\tau)} \to 0$ and identifying the power of t with the specific heat exponent α_τ, we find:

$$\alpha_\tau = 2 - \tau(d - \frac{m}{2})\nu_\tau. \tag{45}$$

Let us analyse the inhomogeneous part. First, take $Q_{(\tau)} = 0$. Second, choose a particular solution of the form:

$$C_p(u_\tau) = (\kappa_\tau^{2\tau})^{\frac{-\epsilon_L}{2}} \tilde{C}_p(u_\tau). \tag{46}$$

Replace this into the RG equation for $\Gamma_{R(\tau)}^{(0,2)}$ at the fixed point. Then, it is easy to obtain

$$C_p(u_\tau^*) = (\kappa_\tau^{2\tau})^{\frac{-\epsilon_L}{2}} \frac{\nu_\tau}{\nu_\tau\tau(d-\frac{m}{2}) - 2} B_\tau(u_\tau^*). \tag{47}$$

Collecting both terms we have the following general solution at the fixed point:

$$\Gamma_{R(\tau)}^{(0,2)} = (\kappa_\tau^{-2\tau})^{\frac{\epsilon_L}{2}}(C_\tau(\frac{t}{\kappa_\tau^{2\tau}})^{-\alpha_\tau} + \frac{\nu_\tau}{\nu_\tau\tau(d-\frac{m}{2}) - 2} B_\tau(u_\tau^*)). \tag{48}$$

Let us describe the situation for $T < T_L$. It can be illustrated for the case of magnetic systems. The renormalized equation of state relates the renormalized (1PI one-point vertex part) magnetic field with the renormalized vertex parts for $t < 0$ via a power series in the magnetization M. One has:

$$H_{(\tau)}(t,M,u_\tau,\kappa_\tau) = \sum_{N=1}^{\infty} \frac{1}{N!} M^N \Gamma^{1+N}_{R(\tau)}(k_{i(\tau)}=0;t,u_\tau,\kappa_\tau), \tag{49}$$

where the zero momentum limit must be taken after realizing the summation. The magnetic field satisfies the following RG equation:

$$(\kappa_\tau \frac{\partial}{\partial \kappa_\tau} + \beta_\tau \frac{\partial}{\partial u_\tau} - \frac{1}{2} N \gamma_{\phi(\tau)}(N + M\frac{\partial}{\partial M}) + \gamma_{\phi^2(\tau)} t \frac{\partial}{\partial t}) H_{(\tau)}(t,M,u_\tau,\kappa_\tau) = 0. \tag{50}$$

At the fixed point we have the following form for the equation of state:

$$H_{(\tau)}(t,M,\kappa_1) = \kappa_\tau^{\frac{\eta_\tau}{2}} h_{1\tau}(\kappa_\tau M^{\frac{2}{\eta_\tau}}, \kappa_\tau t^{\frac{-1}{\gamma_{\phi^2(\tau)}}}). \tag{51}$$

Once again, we use dimensional analysis arguments to obtain the following expression under a flow in the momenta :

$$H_{(\tau)}(t,M,\kappa_\tau) = \rho_\tau^{\tau[\frac{d-\frac{m}{2}}{2}+1]} H_{(\tau)}(\frac{t}{\rho_\tau^{2\tau}}, \frac{M}{\rho_\tau^{2\tau[\frac{d-\frac{m}{2}}{2}-1]}}, \frac{\kappa_\tau}{\rho_\tau}). \tag{52}$$

We choose ρ_τ to be a power of M such that:

$$\rho_\tau = \kappa_\tau [\frac{M}{\kappa_\tau^{\frac{\tau}{2}[(d-\frac{m}{2})-2]}}]^{\frac{2}{\tau[(d-\frac{m}{2})-2]+\eta_\tau}}, \tag{53}$$

and from the scaling form of the equation of state $H_{(\tau)}(t,M) = M^{\delta_\tau} f(\frac{t}{M^{\frac{1}{\beta_\tau}}})$, we obtain the remaining scaling laws after the identifications $\delta_1 = \delta_{L2}, \beta_1 = \beta_{L2}$, $\delta_2 = \delta_{L4}, \beta_2 = \beta_{L4}$, :

$$\delta_{L2} = \frac{(d-\frac{m}{2})+2-\eta_{L2}}{(d-\frac{m}{2})-2+\eta_{L2}}, \tag{54a}$$

$$\beta_{L2} = \frac{1}{2}\nu_{L2}((d-\frac{m}{2})-2+\eta_{L2}), \tag{54b}$$

$$\delta_{L4} = \frac{2(d-\frac{m}{2})+4-\eta_{L4}}{2(d-\frac{m}{2})-4+\eta_{L4}}, \tag{54c}$$

$$\beta_{L4} = \frac{1}{2}\nu_{L4}(2(d-\frac{m}{2})-4+\eta_{L4}), \tag{54d}$$

$$\tag{54e}$$

which imply the Widom $\gamma_{L2} = \beta_{L2}(\delta_{L2}-1)$ and Rushbrook $\alpha_{L2} + 2\beta_{L2} + \gamma_{L2} = 2$ relations for directions perpendicular to the competition axes. These relations are also valid for directions along the competing axes. So far, the effect of considering this new dimensional role played by the momenta scale along the competing quartic subspace, together with the definitions of the critical theories either at vanishing quartic or quadratic external momenta have induced two independent set of scaling relations for the critical exponents. Nevertheless, when performing the diagramatic perturbative expansion, we shall find out that some of these exponents are not independent.

There is one curious fact relating these scaling relations and those first obtained by Hornreich et al. [1]. If $\nu_{L4} = \frac{1}{2}\nu_{L2}$ and $\eta_{L4} = 2\eta_{L2}$ to all orders in perturbation theory, the

hyperscaling (Josephson) relations are the same in either formulation. In the formulation presented here this feature will be found in Sec. VI from the perturbative analysis up to the loop order considered in this paper. In addition, one obtains

$$\gamma_{L4} = \gamma_{L2} = \gamma_L, \tag{55a}$$

$$\beta_{L4} = \beta_{L2} = \beta_L, \tag{55b}$$

$$\delta_{L4} = \delta_{L2} = \delta_L, \tag{55c}$$

$$\alpha_{L4} = \alpha_{L2} = \alpha_L. \tag{55d}$$

In that case, there is a complete equivalence among the scaling relations in both formulations. It is worth to emphasize that the advantage of the approach presented in this section is the splitting of the scaling laws into independent renormalization group flows parallel and perpendicular to the competition axes. Then, instead of claiming that the two momenta scales corresponding to the components perpendicular and parallel to the competing axes are equal [1], the most important conclusion in our approach with two coupling constants is that they will flow to the same fixed point in the critical regime as will be shown later.

We can make a comparison among our scaling relations below T_L with the ones obtained by Mergulhão and Carneiro. In their work, they defined the space dimension of the Lifshitz system $D \equiv \frac{d_\alpha}{2} + d_\beta$, where $d_\alpha = m$ and $d_\beta = d - m$ (see Eq.(25) in [28]). Therefore, $D = (d - \frac{m}{2})$, the same effective space dimension as ours.

The trouble is in the introduction of σ. The normalization conditions defined through Eqs.(20)-(24) of Ref. [28] mix σ with the two external momenta scales in a nontrivial way. They recognized, however, that the normalization conditions they defined are *independent* of σ. Thus if one makes the choice $\gamma_\sigma^* = 0$, Eq. (58) in Ref. [28] is just the same as the one obtained here for δ_L, and Eq.(60) in Ref. [28] is equal to that obtained here for β_L. The Eqs. (46)-(51) in their article taken together with $\gamma_\sigma^* = 0$ yields trivially $\nu_{L4} = \frac{1}{2}\nu_{L2}$ and $\eta_{L4} = 2\eta_{L2}$. Last but not least, if one takes the bare σ_0 dimensionless, as was done by those authors, the whole argument is invalidated since its *dimensionfulness* was assured from the beginning of the discussion in the regulation of the free critical propagator. Introducing σ is consistent provided it is considered as a dimensionful ammount in all stages of the calculation. In other words, σ is not required at all, since the flow in σ can be absorbed in the quartic momenta scale using our dimensional redefinition.

IV. RENORMALIZATION GROUP FOR THE ISOTROPIC BEHAVIOR

The procedure to analyse the isotropic case is quite analogous. Some care must be taken. Whenever τ appears as a subscript, like in a quantity A_τ, one sets $\tau = 3$ in order to be consistent with the notation employed so far. The dimensional analysis is a bit different. The volume element in momentum space is again $d^{d-m}qd^mk$. Whenever $d = m$, the volume element is now d^mk. As before, $[k] = M^{\frac{1}{2}}$. Accordingly, the volume element has dimension $[d^mk] = M^{\frac{m}{2}}$. The dimension of the field in mass units is $[\phi] = M^{\frac{m}{4}-1}$. When the conserving δ-function is removed the 1PI vertex parts have dimensions $[\Gamma^{(N)}(k_i)] = M^{N+\frac{m}{2}-N\frac{m}{4}}$. Then, make the continuation $m = 8-\epsilon_L$. The coupling constant has dimension $\lambda_3 = M^{\frac{8-m}{2}} = M^{\frac{\epsilon_L}{2}}$. In terms of dimensionless quantities, one has the renormalized $g_3 = u_3(\kappa_3^4)^{\frac{\epsilon_L}{4}}$ and bare $\lambda_3 = u_{03}(\kappa_3^4)^{\frac{\epsilon_L}{4}}$ coupling constants, respectively. Again, the functions

$$\beta_3 = (\kappa_3 \frac{\partial u_3}{\partial \kappa_3}) \tag{56a}$$

$$\gamma_{\phi(3)}(u_3) = \beta_3 \frac{\partial ln Z_{\phi(3)}}{\partial u_3} \tag{56b}$$

$$\gamma_{\phi^2(3)}(u_3) = -\beta_3 \frac{\partial ln Z_{\phi^2(3)}}{\partial u_3} \tag{56c}$$

are calculated, as before, at fixed bare coupling λ_3. The beta function can be expressed in terms of dimensionless quantities as $\beta_3 = -\epsilon_L (\frac{\partial ln u_{03}}{\partial u_3})^{-1}$. One should notice that the beta function for the isotropic case does not possess the overall factor of 2 present in the anisotropic beta function β_2 obtained from renormalization group transformations over the quartic momenta scale κ_2. This feature is a strong nonperturbative suggestion that the isotropic critical observables cannot be obtained from the anisotropic ones and vice-versa.

Under a flow in the quartic momenta, from our dimensional analysis, the dimensional redefinition of the momenta along the competing axes results in an effective space dimension for the isotropic case, i.e., $(\frac{m}{2})$. Thus, we find the following behavior for the 1PI vertex parts $\Gamma_{R(3)}^{(N)}$ under a flow in the external momenta:

$$\begin{aligned}\Gamma_{R(3)}^{(N)}(\rho_3 k_i, u_3, \kappa_3) = \rho_3^{2[N+\frac{m}{2}-N\frac{m}{4}]} exp[-\tfrac{N}{2}\textstyle\int_1^{\rho_3} \gamma_{\phi(3)}(u_3(\rho_3))\frac{dx_3}{x_3}] \\ \times\ \Gamma_{R(3)}^{(N)}(k_i, u_3(\rho_3), \kappa_3\rho_3).\end{aligned} \tag{57}$$

Notice that we put aside the notation $k_{i(\tau)}$, etc., used in the anisotropic analysis in favor of k_i, etc. since there is only one quartic momenta scale in the isotropic case. At the fixed point, we also have a simple scaling property for the vertex parts $\Gamma_{R(3)}^{(N)}$, namely:

$$\begin{aligned}\Gamma_{R(3)}^{(N)}(\rho_3 k_i, u_3^*, \kappa_3) = \rho_3^{2[N+\frac{m}{2}-N\frac{m}{4}]-\frac{N}{2}\gamma_{\phi(3)}(u_3^*)} \\ \times \Gamma_{R(3)}^{(N)}(k_i, u_3^*, \kappa_3).\end{aligned} \tag{58}$$

For $N = 2$, we have

$$\Gamma_{R(3)}^{(2)}(\rho_3 k, u_3^*, \kappa_3) = \rho_3^{4-\gamma_{\phi(3)}(u_3^*)} \Gamma_{R(3)}^{(2)}(k, u_3^*, \kappa_3). \tag{59}$$

We can now identify $\eta_{L4} \equiv \eta_3 = \gamma_{\phi(3)}(u_3^*)$ as the anomalous dimension for the isotropic case. This is the analogue of the analysis we performed for the anisotropic case. In the free theory $d_\phi^0 = \frac{\frac{m}{2}}{2} - 1$ is the naive dimension of the field. At the isotropic fixed point, the naive dimension is modified due to the presence of interactions, such that $d_\phi = \frac{\frac{m}{2}}{2} - 1 + \frac{\eta_{L4}}{4}$. The generalization to include L insertions of ϕ^2 operators is quite straightforward and can be written at the fixed point as $((N, L) \neq (0, 2))$:

$$\Gamma_{R(3)}^{(N,L)}(\rho_3 k_i, \rho_3 p_i, u_3^*, \kappa_3) = \rho_3^{2[N+\frac{m}{2}-\frac{N(\frac{m}{2})}{2}-2L]-\frac{N\gamma_{\phi(3)}^*}{2}+L\gamma_{\phi^2(3)}^*} \Gamma_{R(3)}^{(N,L)}(k_i, p_i, u_3^*, \kappa_3). \tag{60}$$

Thus, if we write at the fixed point

$$\Gamma_{R(3)}^{(N,L)}(\rho_3 k_i, \rho_3 p_i, u_3^*, \kappa_3) = \rho_3^{\frac{m}{2}-Nd_\phi+Ld_{\phi^2}} \Gamma_{R(3)}^{(N,L)}(k_i, p_i, u_3^*, \kappa_3), \tag{61}$$

the anomalous dimension of the insertions of ϕ^2 operators is given by $d_{\phi^2} = -4 + \gamma_{\phi^2(3)}(u_3^*)$.

Above the Lifshitz critical temperature, the renormalized vertex parts for $t \neq 0$ can be expanded as a power series in t around those renormalized vertices at $t = 0$, provided $N \neq 0$. We can now apply the differential operators

$$O_3 = \kappa_3 \frac{\partial}{\partial \kappa_3} + \beta_3 \frac{\partial}{\partial u_3} - \frac{1}{2} N \gamma_{\phi(3)}(u_3) + \gamma_{\phi^2(3)}(u_3) t \frac{\partial}{\partial t} \tag{62}$$

to $\Gamma^{(N)}_{R(3)}(k_i, t, u_3^*, \kappa_3)$. The mechanism is similar to that discussed in the anisotropic cases. This vertex part is a power series on t, with each individual coefficient vanishing by making use of the RGE for $\Gamma^{(N,L)}_{R(3)}(k_i, p_i, u_3^*, \kappa_3)$. Then, we find

$$[\kappa_3 \frac{\partial}{\partial \kappa_3} + \beta_3 \frac{\partial}{\partial u_3} - \frac{1}{2} N \gamma_{\phi(3)}(u_3) + \gamma_{\phi^2(3)}(u_3) t \frac{\partial}{\partial t}] \Gamma^{(N)}_{R(3)}(k_i, t, u_3^*, \kappa_3) = 0. \tag{63}$$

The solution is a homogeneous function of the product of k_i (to some power) and t only at the fixed point u_3^*. The solution reads:

$$\Gamma^{(N)}_{R(3)}(k_{i(3)}, t, u_3^*, \kappa_3) = \kappa_3^{\frac{N\gamma^*_{\phi(3)}}{2}} F^{(N)}_{(3)}(k_i, \kappa_3 t^{\frac{-1}{\gamma^*_{\phi^2(3)}}}). \tag{64}$$

All the exponents generated by the renormalization group flow along the scale κ_3 will be denoted by a corresponding $L4$ subscript. If we define $\theta_3 = -\gamma^*_{\phi^2(3)}$, one can use dimensional analysis to obtain

$$\begin{aligned} \Gamma^{(N)}_{R(3)}(k_i, t, \kappa_3) &= \rho_3^{2[N + \frac{m}{2} - \frac{N}{2}\frac{m}{2}] - \frac{N}{2}\eta_{L4}} \kappa_3^{\frac{N}{2}\eta_{L4}} \\ &\times \ F^{(N)}_{(3)}(\rho_3^{-1} k_i, (\rho_3^{-1}\kappa_3)(\rho_3^{-4} t)^{\frac{1}{\theta_3}}). \end{aligned} \tag{65}$$

One can choose $\rho_3 = \kappa_3 (\frac{t}{\kappa_3^4})^{\frac{1}{\theta_3 + 4}}$, and replacing it in (64), the vertex function depends only on the combination $k_i \xi_{L4}$ apart from a power of t. As ξ_{L4} is proportional to $t^{-\nu_{L4}}$ the critical exponent $\nu_3 = \nu_{L4}$ can be identified as

$$\nu_{L4}^{-1} = 4 + \theta_3^* = 4 - \gamma^*_{\phi^2(3)}. \tag{66}$$

Again it is convenient to define the function

$$\bar{\gamma}_{\phi^2(3)}(u_3) = -\beta_3 \frac{\partial ln(Z_{\phi^2(3)} Z_{\phi(3)})}{\partial u_3}. \tag{67}$$

Then, one can easily find the following relation

$$\nu_{L4}^{-1} = 4 - \eta_{L4} - \bar{\gamma}_{\phi^2(3)}(u_3^*). \tag{68}$$

At the fixed point all correlation functions (not including composite operators) scale at $T > T_L$, since they are functions of $k_i \xi_{L4}$ only. For $N = 2$ we choose $\rho_3 = k$, the external momenta. The two-point vertex part can be written in the form $\Gamma^{(2)}_{R(3)}(k, t, \kappa_3) = k^{4-\eta_{L4}} \kappa_3^{\eta_{L4}} f(k\xi_{L4})$. The main point is that when $\xi_{L4} \to \infty$ and $k \to 0$, simultaneously, then $f(k\xi_{L4}) \to Constant$. Defining $f_3 = (k\xi_{L4})^{4-\eta_{L4}} f(k\xi_{L4})$, we have $\Gamma^{(2)}_{R(3)}(k, t, \kappa_3) = (k\xi_{L4})^{4-\eta_{L4}} \kappa_3^{\eta_{L4}} f_3(k\xi_{L4})$. The

susceptibility is proportional to $t^{-\gamma_{L4}}$ as $k_i \rightarrow 0$. Since $\Gamma_R^{(2)} = \chi^{-1}$, the susceptibility critical exponent is given by

$$\gamma_{L4} = \nu_{L4}(4 - \eta_{L4}). \tag{69}$$

The scaling relation for the specific heat exponent can be found from the RG equation for $\Gamma_{R(3)}^{(0,2)}$ above T_L at the fixed point, namely

$$(\kappa_3 \frac{\partial}{\partial \kappa_3} + \gamma_{\phi^2(3)}^*(2 + t\frac{\partial}{\partial t}))\Gamma_{R(3)}^{(0,2)} = (\kappa_3^{-2})^{\frac{\epsilon_L}{2}} B_3(u_3^*), \tag{70}$$

where

$$(\kappa_3^{-4})^{\frac{\epsilon_L}{4}} B_3(u_3^*) = -Z_{\phi^2(3)}^2 \kappa_3 \frac{\partial}{\partial \kappa_3} \Gamma_{(3)}^{(0,2)}(Q; -Q, \lambda_3), \tag{71}$$

is the inhomogeneous part which does not depend on t. Recall that the bare vertex function $\Gamma_{(3)}^{(0,2)}$ is calculated as before in the limit $\Lambda_3 \rightarrow \infty$, with a fixed bare coupling constant, which renders $B_3(u_3^*)$ finite in this limit when $m = 8$. This renormalized vertex part is made out of the addition of the homogeneous (temperature dependent) and inhomogeneous pieces. The general discussion given so far for the vertex part $\Gamma_{R(3)}^{(N,L)}$ is helpful to obtain the homogeneous part of the solution. At the fixed point we have the following generalization of the solution for $\Gamma_{R(3)}^{(N,L)}$:

$$\Gamma_{R(3)}^{(N,L)}(p_i, Q_i, t, \kappa_3) = \kappa_3^{\frac{1}{2}N\gamma_{\phi(3)}^* - L\gamma_{\phi^2(3)}^*} F_3^{(N,L)}(p_i, Q_i, \kappa_3 t^{\frac{-1}{\gamma_{\phi^2(3)}^*}}). \tag{72}$$

The temperature dependent homogeneous part for $\Gamma_{R(3),h}^{(0,2)}$ scales at the fixed point, i.e.,

$$\Gamma_{R(3),h}^{(0,2)}(Q, -Q, t, \kappa_3) = \kappa_3^{-2\gamma_{\phi^2(3)}^*} F_3^{(0,2)}(Q, -Q, \kappa_3 t^{\frac{-1}{\gamma_{\phi^2(3)}^*}}). \tag{73}$$

This vertex function is to be identified with the specific heat at zero external momentum insertion $Q = 0$. Using our dimensional analysis one finds:

$$\Gamma_{R(3),h}^{(0,2)}(Q, -Q, t, \kappa_3) = \rho^{2[\frac{m}{2} - 4] + 2\gamma_{\phi^2(3)}^*} \Gamma_{R(3),h}^{(0,2)}(\rho_3^{-1}Q, -\rho_3^{-1}Q, \rho_3^{-4}t, \rho_3^{-1}\kappa_3). \tag{74}$$

Substitution of this equation into the solution at the fixed point yields

$$\Gamma_{R(3),h}^{(0,2)}(Q, -Q, t, \kappa_3) = \rho_3^{2[\frac{m}{2} - 4] + 2\gamma_{\phi^2(3)}^*} \kappa_3^{-2\gamma_{\phi^2(3)}^*} F_3^{(0,2)}(\rho_3^{-1}Q, -\rho_3^{-1}Q, \rho_3^{-1}\kappa_3(\rho_3^{-4}t)^{\frac{-1}{\gamma_{\phi^2(3)}^*}}). \tag{75}$$

We make the choice $\rho_3 = \kappa_3(\frac{t}{\kappa_3^4})^{\frac{1}{\theta_3 + 4}}$. Replacing this into the last equation, taking the limit $Q \rightarrow 0$ and identifying the power of t with the specific heat exponent α_{L4}, we find:

$$\alpha_{L4} = 2 - m\nu_{L4}. \tag{76}$$

The description of the inhomogeneous part is as follows. First, take $Q = 0$. Then, choose a particular solution of the form:

$$C_p(u_3) = (\kappa_3^4)^{\frac{-\epsilon_L}{4}} \tilde{C}_p(u_3). \qquad (77)$$

Now replace this into the RG equation for $\Gamma_{R(3)}^{(0,2)}$ at the fixed point. Therefore, one gets to

$$C_p(u_3^*) = (\kappa_3^4)^{\frac{-\epsilon_L}{4}} \frac{\nu_{L4}}{\nu_{L4} m - 2} B_3(u_3^*)). \qquad (78)$$

The general solution at the fixed point is just the sum of the two pieces, and is given by

$$\Gamma_{R(3)}^{(0,2)} = (\kappa_3^{-4})^{\frac{\epsilon_L}{4}} (C_3(\frac{t}{\kappa_3^4})^{-\alpha_{L4}} + \frac{\nu_{L4}}{\nu_{L4} m - 2} B_3(u_3^*)). \qquad (79)$$

We now turn our attention to analyse the scaling relations when the system is below the Lifshitz critical temperature $T < T_L$. The renormalized magnetic field is related to the renormalized vertex parts for $t < 0$ and the magnetization M through

$$H_{(3)}(t, M, u_3, \kappa_3) = \sum_{N=1}^{\infty} \frac{1}{N!} M^N \Gamma_{R(3)}^{1+N}(k_i = 0; t, u_3, \kappa_3), \qquad (80)$$

where the zero momentum limit must be taken after realizing the summation. The magnetic field satisfies the RG equation:

$$(\kappa_3 \frac{\partial}{\partial \kappa_3} + \beta_3 \frac{\partial}{\partial u_3} - \frac{1}{2} N \gamma_{\phi(3)}(u_3)(N + M \frac{\partial}{\partial M}) + \gamma_{\phi^2(3)} t \frac{\partial}{\partial t}) H_{(3)}(t, M, u_1, \kappa_1) = 0. \qquad (81)$$

The equation of state at the fixed point reads:

$$H_{(3)}(t, M, \kappa_3) = \kappa_3^{\frac{\eta_{L4}}{2}} h_3(\kappa_3 M^{\frac{2}{\eta_{L4}}}, \kappa_1 t^{\frac{-1}{\gamma_{\phi^2(3)}}}). \qquad (82)$$

Dimensional analysis arguments lead to the following expression under a flow in the external momenta:

$$H_{(3)}(t, M, \kappa_3) = \rho_3^{2[\frac{m}{4}+1]} \; H_3(\frac{t}{\rho_3^4}, \frac{M}{\rho_3^{2[\frac{m}{4}-1]}}, \frac{\kappa_3}{\rho_3}). \qquad (83)$$

The flow parameter ρ_3 is chosen to be a power of M such that:

$$\rho_3 = \kappa_3 [\frac{M}{\kappa_3^{[\frac{m}{2}-2]}}]^{\frac{2}{m-4+\eta_{L4}}}, \qquad (84)$$

and from the scaling form of the equation of state $H_{(3)}(t, M) = M^{\delta_{L4}} f(\frac{t}{M^{\frac{1}{\beta_{L4}}}})$, we obtain the following scaling laws:

$$\delta_{L4} = \frac{m + 4 - \eta_{L4}}{m - 4 + \eta_{L4}}, \qquad (85a)$$

$$\beta_{L4} = \frac{1}{2} \nu_{L4} (m - 4 + \eta_{L4}), \qquad (85b)$$

which imply the Widom $\gamma_{L4} = \beta_{L4}(\delta_{L4} - 1)$ and Rushbrook $\alpha_{L4} + 2\beta_{L4} + \gamma_{L4} = 2$ relations.

The scaling relations for the anisotropic case Eqs.(11a),(12) in Ref. [1] for $d = m$ are consistent with the isotropic case. Note, however, that this cannot be given a rigorous meaning, for the appearance of ν_{L2} and η_{L2} in the equality in Eq.(11b) of Ref. [1] invalidates the argument for the isotropic case as these exponents are no longer meaningful. Notice that the impossibility of finding scaling relations for the isotropic case in the original framework [1] is due to the lack of the independent flow in the external momenta scale κ_3 along the quartic subspace. In the early treatment [1], the quartic momenta was not independent to be varied freely, but was fixed from the variation of the quadratic scale. Without its free variation, which is possible since this quartic term in the propagator does not have the same canonical dimension as the quadratic one, no renormalization group flows along the competing directions can be defined whatsoever. Thus, this new renormalization group method permits to go further in determining the Lifshitz critical universal properties of the system for arbitrary m.

V. THE EVALUATION OF FEYNMAN INTEGRALS

In order to calculate universal quantities like critical exponents, we must calculate some Feynman integrals. The perturbative loop expansion shall be our starting point with the $\epsilon_L = 4 + \frac{m}{2} - d$ being the perturbation parameter for the anisotropic situation. For the isotropic case, the perturbation parameter is $\epsilon_L = 8 - m$.

We have to express the solution of the Feynman diagrams in terms of ϵ_L, resulting in the ϵ_L-expansion for the universal critical ammounts. Again, there is also a very important difference among the anisotropic and isotropic behaviors. From a technical viewpoint, the anisotropic behaviors present two types of integration along the two momenta subspaces, whereas in the isotropic situation there is only one subspace to be integrated over. We shall treat them separately.

The anisotropic behavior is described using two different approximations. We shall briefly discuss the first analytical approximation developed for evaluating higher-order Feynman diagrams which are needed in the calculation of the critical exponents perpendicular to the competing axes for the anisotropic Lifshitz behavior. It points out the necessity of some sort of condition among the quartic loop momenta in different subdiagrams, leading to the homogeneity of the integrals in the quadratic external momenta scales. We employ the set of normalization conditions with vanishing quartic external momenta as described in Sec.II. This piece of work was done in collaboration with L.C. de Albuquerque and the details can be found in [20, 23].

Nevertheless, with the renormalization group description presented here, this approximation is not sufficient to describe the critical exponents along the competing axes. It does not yield the solution of the integrals as a homogeneous function of *both* quadratic and quartic external momenta scales yet. The former approximation described above is then generalized to obtain the solution of the integrals for *arbitrary* quadratic and quartic external momenta scales. Using the new interpretation for the momenta scale along the quartic direction given in the last two sections, the calculation of these integrals is not a complicated task, provided a certain condition among the quartic momenta is fulfilled. With this new technique all the critical exponents in the anisotropic cases are obtained. This picture can be considered the main result of the present work.

The isotropic behavior can be developed along the same lines of the latter approach to the anisotropic case. The condition among the quartic momenta is also required in order to guarantee homogeneity of the Feynman integrals in the quartic external momenta scale. The new approximation is sufficient to complete the unified analytical description of the Lifshitz critical behavior in its full generality, at least at the loop order considered here as will be shown in this section.

In order to verify the renormalization scheme independence of the critical exponents, it would be interesting to obtain the critical exponents using more than one renormalization procedure. In fact, as will be proven later, the use of normalization conditions or minimal subtraction of dimensional poles yield the same critical exponents. Thus, we shall present the results in the most appropriate form for calculating the critical exponents in these two renormalization prescriptions.

A. Anisotropic

In order to calculate universal quantities like critical exponents, we must calculate some Feynman integrals. We start by listing all the relevant integrals which are necessary to find out the critical exponents. These integrals are

$$I_2 = \int \frac{d^{d-m}q d^m k}{\left[((k+K')^2)^2 + (q+P)^2\right]\left((k^2)^2 + q^2\right)} \quad , \tag{86}$$

where I_2 is the one-loop integral contributing to the four-point function,

$$I_3 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{\left(q_1^2 + (k_1^2)^2\right)\left(q_2^2 + (k_2^2)^2\right)\left[(q_1+q_2+P)^2 + ((k_1+k_2+K')^2)^2\right]} \quad , \tag{87}$$

is the two-loop "sunset" Feynman diagram of the two-point function,

$$\begin{aligned} I_4 &= \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{\left(q_1^2 + (k_1^2)^2\right)\left((P-q_1)^2 + ((K'-k_1)^2)^2\right)\left(q_2^2 + (k_2^2)^2\right)} \\ &\qquad \times \frac{1}{(q_1-q_2+p_3)^2 + ((k_1-k_2+k_3')^2)^2} \quad . \end{aligned} \tag{88}$$

is one of the two-loop graphs which will contribute to the fixed-point, and

$$\begin{aligned} I_5 &= \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^{d-m}q_3 d^m k_1 d^m k_2 d^m k_3}{\left(q_1^2 + (k_1^2)^2\right)\left(q_2^2 + (k_2^2)^2\right)\left(q_3^2 + (k_3^2)^2\right)\left[(q_1+q_2-p)^2 + ((k_1+k_2-k')^2)^2\right]} \\ &\qquad \times \frac{1}{(q_1+q_3-p)^2 + ((k_1+k_3-k')^2)^2} \end{aligned} \tag{89}$$

is the three-loop diagram contributing to the two-point vertex function. Now we proceed to calculate these integrals using two different approximation schemes. The philosophy to be addopted is to simplify the calculation by making use of the homogeneity hypothesis, as shall become clear in the following subsections.

1. The "dissipative" approximation

As this approximation is only suited to calculate the integral as a function of the quadratic external momenta, we set the external momenta at the quartic directions equal to zero, i.e. $k' = k'_1 = k'_2 = k'_3 = 0$, and $K' = k'_1 + k'_2$. We shall use dimensional regularization for the calculation of the Feynman diagrams.

Let us find out the one-loop integral I_2. With our choice of the symmetry point, and introducing two Schwinger's parameters we obtain for I_2:

$$\int \frac{d^{d-m}q d^m k}{((k^2)^2 + (q+P)^2)\,((k^2)^2 + q^2)} = \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \Big(\int d^m k \, \exp(-(\alpha_1+\alpha_2)(k^2)^2)\Big)$$
$$\times \int d^{d-m}q \, \exp(-(\alpha_1+\alpha_2)q^2 - 2\alpha_2 q.P - \alpha_2 P^2). \tag{90}$$

The $\vec{q}$ integral can be performed to give

$$\int d^{d-m}q \, \exp(-(\alpha_1+\alpha_2)q^2 - 2\alpha_2 q.P - \alpha_2 P^2)$$
$$= \frac{1}{2} S_{d-m}\Gamma(\frac{d-m}{2})(\alpha_1+\alpha_2)^{-\frac{d-m}{2}} \exp(-\frac{\alpha_1\alpha_2 P^2}{\alpha_1+\alpha_2}) \ . \tag{91}$$

For the $\vec{k}$ integral we perform the change of variables $r^2 = k_1^2 + ... + k_m^2$. Now take $z = r^4$. The integral turns out to be:

$$\int d^m k \, \exp(-(\alpha_1+\alpha_2)(k^2)^2) = \Big(\frac{1}{4}S_m\Big)\Gamma(\frac{m}{4})(\alpha_1+\alpha_2)^{-\frac{m}{4}}. \tag{92}$$

Using Eqs. (90) and (91), I_2 reads

$$I_2 = \frac{1}{2} S_{d-m}\Big(\frac{1}{4}S_m\Big)\Gamma(\frac{d-m}{2})\Gamma(\frac{m}{4})$$
$$\times \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \, \exp(-\frac{\alpha_1\alpha_2 P^2}{\alpha_1+\alpha_2}) \, (\alpha_1+\alpha_2)^{-(\frac{d}{2}-\frac{m}{4})}. \tag{93}$$

The remaining parametric integrals can be solved by a change of variables followed by a rescaling [23]. The integral is proportional to $(P^2)^{-\frac{\epsilon_L}{2}}$. Now we can set $P^2 = \kappa^2 = 1$. Using the identity

$$\Gamma(a+bx) = \Gamma(a)\Big[1 + b\,x\,\psi(a) + O(x^2)\Big], \tag{94}$$

where $\psi(z) = \frac{d}{dz} ln\Gamma(z)$, one is able to perform the ϵ_L-expansion when the Gamma functions have non integer arguments. Altogether, the final result for I_2 is:

$$I_2 = \Bigg[\frac{1}{4} S_m S_{d-m} \Gamma(2-\frac{m}{4})\Gamma(\frac{m}{4})\Bigg] \frac{1}{\epsilon_L}\Big(1 + [i_2]_m \, \epsilon_L\Big) \quad , \tag{95}$$

where $[i_2]_m = 1 + \frac{1}{2}(\psi(1) - \psi(2-\frac{m}{4}))$. The factor inside the brackets in Eq. (94) is absorbed in a redefinition of the coupling constant. Then the redefined integral is:

$$I_2 = \frac{1}{\epsilon_L}\Big(1 + [i_2]_m \, \epsilon_L\Big) \quad . \tag{96}$$

Note that this expression involves no approximation. This simple result is a consequence of the absence of the zero quartic external momenta. Had we considered it from the beggining, we would have obtained an intermediate integral that could not be integrated analytically. We shall discuss this issue later in the next subsections.

However, when we go on to calculate higher loop integrals, some sort of approximation is required, since these integrals are complicated by the fact that even with zero external quartic momenta, the quartic loop momenta mix up in different subdiagrams in a extremely non-trivial form. As an example, we discuss I_3. It is given by:

$$I_3(P, K') = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{(q_1^2 + (k_1^2)^2)\,(q_2^2 + (k_2^2)^2)\,[(q_1+q_2+P)^2 + ((k_1+k_2+K')^2)^2]} \quad . \tag{97}$$

Setting $K' = 0$, the integral can be evaluated as outlined in [20]. Before making our approximation, one can choose to integrate first either over the loop momenta (q_1, k_1) or over (q_2, k_2). The loop integrals to be integrated first are referred to as the internal bubbles. By solving the integral over q_2 first, we obtain

$$I_3(p,0) = \frac{1}{2}S_{d-m}\Gamma(\frac{d-m}{2})\int \frac{d^{d-m}q_1 d^m k_1}{q_1^2 + (k_1^2)^2}$$
$$\int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 (\alpha_1+\alpha_2)^{\frac{-(d-m)}{2}} exp(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}(q_1+p)^2)\int d^m k_2 e^{-\alpha_1(k_2^2)^2} e^{-\alpha_2((k_1+k_2)^2)^2}. \tag{98}$$

Now we can consider the approximation. In order to integrate over k_2, we have to expand the argument of the last exponential. This will produce a complicated function of α_1, α_2, k_1 and k_2 which cannot be integrated analytically. Considering the remaining terms as a damping factor to the integrand, the maximum of the integrand will be either at $k_1 = 0$ or at $k_1 = -2k_2$. The most general choice $k_1 = -\alpha k_2$ yields a hypergeometric function. The choice $k_1 = -2k_2$ implies that k_1 varies in the internal bubble, but not arbitrarily. Its variation, however, is dominated by k_2 through this constraint, which eliminates the dependence on k_1 in the internal bubble. At these values of k_1, the integration over k_2 produces a simple factor to the parametric integral proportional to $(\alpha_1+\alpha_2)^{-\frac{m}{4}}$. This allows one to perform the remaining parametric integrals in a simple way. After performing these integrals, they produce the factor $((q_1+P)^2)^{-\frac{\epsilon_L}{2}}$. Note that the diagrams I_3 and I_5 contributing to the two-point function receive the factor $\frac{1}{2-\frac{m}{4}}$ after integrating over the quadratic momenta in the external bubble. This factor will not be present in the isotropic case, since there is no integration over quadratic momenta to be done in this case. The resulting solution to $I_3(P,0)$ is a homogeneous function of the external momenta P, given by:

$$I_3 = -(P^2)^{1-\epsilon_L}\frac{1}{8-m}\frac{1}{\epsilon_L}\Big[1 + \Big([i_2]_m + \frac{3}{4-\frac{m}{2}} + 1\Big)\epsilon_L\Big]. \tag{99}$$

The implementation of this constraint on higher-loop integrals proceeds analogously. The constraint turns all these integrals into homogeneous functions of the external quadratic

momenta scale. One can then choose the symmetry point as $P^2 = \kappa_1^2 = 1$, for example, in order to define the renormalized vertices via normalization conditions.

Using this constraint we can easily find the following results at the symmetry point [20]:

$$I_4 = \frac{1}{2\epsilon_L^2}\Big(1 + 3\,[i_2]_m \epsilon_L\Big). \tag{100}$$

The integrals I_3' and I_5' are given by

$$I_3' = -\frac{1}{8-m}\,\frac{1}{\epsilon_L}\left[1 + \left([i_2]_m + \frac{3}{4-\frac{m}{2}}\right)\epsilon_L\right], \tag{101}$$

$$I_5' = -\frac{1}{3(2-\frac{m}{4})}\frac{1}{\epsilon_L^2}\left[1 + 2\left([i_2]_m + \frac{1}{2-\frac{m}{4}}\right)\epsilon_L\right]. \tag{102}$$

Note that the leading singularities for I_2, I_4 are the same as their analogous integrals in the pure ϕ^4 theory. However, I_3' and I_5' do not have the same leading singularities for they include a factor of $\frac{1}{(2-\frac{m}{4})}$. We then introduce a weight factor for I_3' and I_5', namely $(1-\frac{m}{8})$, so that they have the same leading singularities as in the pure ϕ^4 theory. The main drawback of this approximation is the failure to treat the isotropic case. Furthermore, the introduction of weight factors to the two and three-loop diagrams is rather undesirable. Moreover, the constraint among the quartic loop momenta does not allow loop momentum conservation along the quartic subspace in higher-loop diagrams. It is then appropriate to name this approximation "the dissipative approximation".

It is obvious that some important detail is missing. A proper solution of the Feynman integrals should be expressed as a homogeneous function of both external momenta scales. We proceed to discuss the novel approximation which presents this property.

2. The orthogonal approximation

Before considering the integrals to be performed, let us derive some useful formulas which relate Gamma functions with certain intermediate parametric integrals. They will allow us to define a new analytic dimensional regularization procedure in the competing subspace.

The simple integral

$$\int_0^\infty exp(-ax^\mu)dx = a^{\frac{-1}{\mu}}\frac{1}{\mu}\Gamma(\frac{1}{\mu}), \tag{103}$$

can be generalized to the m-sphere. We shall analyse the case $\mu = 2n$. Take $r^2 = x_1^2 + ... + x_m^2$. After that take $z = r^{2n}$. Thus,

$$\int_{-\infty}^\infty dx_1...dx_m exp(-a(x_1^2 + ... + x_m^2)^n) = \frac{1}{2n}\int_0^\infty dz exp(-az) z^{\frac{m}{2n}-1}\int d\Omega_m. \tag{104}$$

The angular integral will produce the area of the m-dimensional sphere, yielding

$$\int_{-\infty}^\infty dx_1...dx_m exp(-(x_1^2 + ... + x_m^2)^n) = \frac{1}{2n}\Gamma(\frac{m}{2n})a^{\frac{-m}{2n}}S_m. \tag{105}$$

One can write this identity in a different way. After choosing $r^2 = x_1^2 + ... + x_m^2$ take $y = r^2$ and as the integral is given by the expression above, we obtain the intermediate result:

$$\int_0^\infty dy y^{\frac{m}{2}-1} exp(-ay^n) = \frac{1}{n} a^{\frac{-m}{2n}} \Gamma(\frac{m}{2n}). \tag{106}$$

Henceafter we shall keep $n = 2$. The following step is to calculate the integral

$$\int_{-\infty}^\infty exp(-ax^4 - bx^2)dx = 2\int_0^\infty exp(-ax^4 - bx^2)dx. \tag{107}$$

The exact answer is given in terms of a Bessel function of a certain combination of a and b. We wish to pick out only the piece which yields the correct homogenous function of a, i.e., only one term of the series. This can be achieved as follows. Choose $y = x^2$. One obtains

$$\int_{-\infty}^\infty exp(-ax^4 - bx^2)dx = exp(\frac{b^2}{4a}) \int_0^\infty exp(-a(y + \frac{b}{2a})^2) y^{\frac{-1}{2}} dy. \tag{108}$$

We then choose $y' = y + \frac{b}{2a}$ implying that

$$\begin{aligned} \textstyle\int_{-\infty}^\infty exp(-ax^4 - bx^2)dx = exp[\frac{b^2}{4a}][\int_0^\infty exp(-ay'^2)(y' - \frac{b}{2a})^{\frac{-1}{2}} dy' \\ - \textstyle\int_0^{\frac{b}{2a}} exp(-ay'^2)(y' - \frac{b}{2a})^{\frac{-1}{2}} dy']. \end{aligned} \tag{109}$$

Since we are dealing with convergent integrals, we can perform the approximation $(y' - \frac{b}{2a})^{\frac{-1}{2}} = y'^{-\frac{1}{2}} + ...$, and the remaining terms will be subtracted from the last integral, which is a sort of error function. The original integral is then approximated by its leading contribution

$$\int_{-\infty}^\infty exp(-ax^4 - bx^2)dx \cong exp(\frac{b^2}{4a}) \int_0^\infty exp(-ay'^2) y'^{-\frac{1}{2}} dy' = exp(\frac{b^2}{4a}) \frac{1}{2} \Gamma(\frac{1}{4}) a^{-\frac{1}{4}}. \tag{110}$$

It can be shown in a straightforward way that for the m-sphere this result generalizes to

$$\begin{aligned} \textstyle\int_{-\infty}^\infty exp[-a(x_1^2 + ... + x_m^2)^2 - b(x_1^2 + ... + x_m^2)]dx_1...dx_m \cong exp(\frac{b^2}{4a}) S_m \times \\ \textstyle\int_0^\infty exp(-ay'^2) y'^{\frac{m}{2}-1} dy' = exp(\frac{b^2}{4a}) \frac{1}{4} S_m \Gamma(\frac{m}{4}) a^{-\frac{m}{4}}. \end{aligned} \tag{111}$$

We now focus our attention in the integral

$$\int_{-\infty}^\infty \frac{dx_1...dx_m}{[(x_1^2 + ... + x_m^2)^2 + 2a(x_1^2 + ... + x_m^2) + m^2]^\beta}. \tag{112}$$

Take $r^2 = x_1^2 + ... + x_m^2$. Make the change of variables in the radial coordinate $z = r^2$. After that take $z' = z + a$. We then obtain

$$\begin{aligned} \textstyle\int \frac{d^m x}{[(x_1^2+...+x_m^2)^2 + 2a(x_1^2+...+x_m^2) + m^2]^\beta} = \frac{1}{2} S_m \\ \textstyle[\int_0^\infty \frac{(z'-a)^{\frac{m}{2}-1} dz'}{(z'^2 + m'^2)^\beta} - \int_0^a \frac{(z'-a)^{\frac{m}{2}-1} dz'}{(z'^2 + m'^2)^\beta}], \end{aligned} \tag{113}$$

where $m'^2 = m^2 + a^2$. Taking $z'' = z'^2$, expanding the numerator in the first integral, i.e., keeping only the leading term and getting rid of the infinite terms to be subtracted from the second integral, one can write this integral in the approximated form

$$\int \frac{d^m x}{[(x_1^2+...+x_m^2)^2+2a(x_1^2+...+x_m^2)+m^2]^\beta} \cong \frac{1}{4}S_m(m^2-a^2)^{-\beta+\frac{m}{4}} \frac{\Gamma(\frac{m}{4})\Gamma(\beta-\frac{m}{4})}{\Gamma(\beta)}. \qquad (114)$$

We have all ingredients to perform Feynman integrals for arbitrary m. We start by considering the simplest integral, the one-loop integral contributing to the coupling constant, that is,

$$I_2 = \int \frac{d^{d-m}q d^m k}{[((k+K^{'})^2)^2 + (q+P)^2]\,((k^2)^2+q^2)} \quad . \qquad (115)$$

We can use two Schwinger parameters and integrate over the quadratic momenta. Using the formula derived above

$$\int \exp(-p^2) d^d q = \frac{1}{2} S_d \Gamma(\frac{d}{2}), \qquad (116)$$

we obtain

$$I_2 = \frac{1}{2}S_{d-m}\Gamma(\frac{d-m}{2}) \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \, \exp(-\frac{\alpha_1\alpha_2 P^2}{\alpha_1+\alpha_2})$$
$$\times(\alpha_1+\alpha_2)^{-(\frac{d-m}{2})} \int d^m k \exp(-\alpha_1(k^2)^2 - \alpha_2((k+K^{'})^2)^2). \qquad (117)$$

We can now expand the argument of the last exponential. This integral cannot be performed analytically. We are interested in the solution of this integral in a form that preserves homogeneity in both external momenta. Some simplifying condition should be tried to achieve this goal.

The most general approximation to calculate this type of integral which is homogeneous in the external momenta scales can be understood as follows. In the first place, if we set $k.K^{'} = 0$ inside the integral, that has the virtue of eliminating odd powers of the quartic external momenta. Thus, the integral becomes:

$$\int d^m k \exp(-\alpha_1(k^2)^2 - \alpha_2((k+K^{'})^2)^2) = \int d^m k$$
$$\exp(-(\alpha_1+\alpha_2)(k^2)^2 - 2\alpha_2 k^2 (K')^2 - \alpha_2((K^{'})^2)^2). \qquad (118)$$

Using Eq.(111), we have for the last quartic momenta integral:

$$\int d^m k \exp(-\alpha_1(k^2)^2 - \alpha_2((k+K^{'})^2)^2) = S_m \frac{1}{4}\Gamma(\frac{m}{4})(\alpha_1+\alpha_2)^{-\frac{m}{4}}$$
$$\times \; \exp(-\frac{\alpha_1\alpha_2((K^{'})^2)^2}{\alpha_1+\alpha_2}). \qquad (119)$$

We can then express the integral in the following form:

$$I_2 = \frac{1}{8}S_{d-m}S_m\Gamma(\frac{d-m}{2})\Gamma(\frac{m}{4})$$
$$\times \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \, \exp(-\frac{\alpha_1\alpha_2(P^2+((K')^2)^2}{\alpha_1+\alpha_2}) \, (\alpha_1+\alpha_2)^{-(\frac{d}{2}-\frac{m}{4})}. \qquad (120)$$

Take $x = \alpha_1(P^2 + (K'^2)^2)$ and $y = \alpha_2(P^2 + (K'^2)^2)$. After that, define $v = \frac{x}{x+y}$. Thus, the parametric integrals can be done easily by this change of variables. Then, use the identity

$$\Gamma(a+bx) = \Gamma(a)\Big[1 + b\,x\,\psi(a) + O(x^2)\Big], \tag{121}$$

where $\psi(z) = \frac{d}{dz} ln\Gamma(z)$. This will result in the following expression for I_2:

$$\begin{aligned} I_2 = \tfrac{1}{2}[\tfrac{1}{4}S_{(d-m)}S_m\Gamma(2-\tfrac{m}{4})\Gamma(\tfrac{m}{4})](1-\tfrac{\epsilon_L}{2}\psi(2-\tfrac{m}{4}))\Gamma(\tfrac{\epsilon_L}{2}) \\ \times \; \int_0^1 dv(v(1-v)(P^2+((K')^2)^2))^{\frac{-\epsilon_L}{2}}. \end{aligned} \tag{122}$$

This is a homogeneous function (with the same homogeneity degree) in (P, K') just as advertised. But this is not the answer yet. The factor $[\frac{1}{4}S_{(d-m)}S_{\frac{m}{2}}\Gamma(2-\frac{m}{4})\Gamma(\frac{m}{4})]$ can be absorbed in a redefinition of the coupling constant. Hence, we shall absorb exactly this factor after performing each loop integral. Furthermore, the last integral can be expanded as

$$\int_0^1 dv(v(1-v)(P^2+((K')^2)^2))^{\frac{-\epsilon_L}{2}} = 1 - \frac{\epsilon_L}{2}L(P,K'), \tag{123}$$

where

$$L(P,K') = \int_0^1 dv \;\; ln[v(1-v)(P^2+((K')^2)^2)]. \tag{124}$$

Thus, we find the following result for this integral:

$$I_2(P,K') = \frac{1}{\epsilon_L}\Big(1 + ([i_2]_m - 1)\epsilon_L - \frac{\epsilon_L}{2}L(P,K')\Big) \;. \tag{125}$$

This is the form suitable for renormalizing using minimal subtraction. On the other hand, for normalization conditions one has:

$$I_{2SP_1} = I_{2SP_2} = \frac{1}{\epsilon_L}\Big(1 + [i_2]_m\,\epsilon_L\Big) \;, \tag{126}$$

since $L(SP_1 = SP_2) = -2$, with $SP_1 \equiv (P^2 = 1, K' = 0)$ and $SP_2 \equiv (P = 0, (K')^2 = 1)$. When we calculated $I_2(P, K' = 0)$ in the last subsection, the orthogonality condition $k.K' = 0$ between the loop momenta and the external momenta along the quartic subspace was trivial. In the calculation of $I_2(P, K')$, the orthogonality condition allowed the solution to this integral with the correct homogeneous properties in both external momenta scales.

We can now turn our attention to the higher-loop integrals. The simplifying condition $k.K' = 0$ for the one-loop integral can be easily generalized to the higher-loop integrals by stating that *the loop momenta in a given bubble (subdiagram) is orthogonal to all loop momenta not belonging to that bubble*. Let us see how this works in the calculation of the "sunset" two-loop integral I_3 contributing to the two-point function, given by the following expression:

$$I_3(P,K') = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{(q_1^2+(k_1^2)^2)\,(q_2^2+(k_2^2)^2)\,[(q_1+q_2+P)^2+((k_1+k_2+K')^2)^2]} \;. \tag{127}$$

We can choose to integrate first over the loop momenta (q_1, k_1) or over (q_2, k_2). The loop integrals to be integrated first are referred to as the internal bubbles. By solving the integral over q_2 first, we obtain

$$I_3(P,K') = \frac{1}{2}S_{d-m}\Gamma(\frac{d-m}{2})\int \frac{d^{d-m}q_1 d^m k_1}{q_1^2+(k_1^2)^2}\int_0^{\infty}\int_0^{\infty} d\alpha_1 d\alpha_2 (\alpha_1+\alpha_2)^{\frac{-(d-m)}{2}}$$
$$exp(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}(q_1+P)^2)\int d^m k_2 e^{-\alpha_1 (k_2^2)^2} e^{-\alpha_2((k_1+k_2+K')^2)^2}. \tag{128}$$

We define $K'' = k_1 + K'$ into the argument of the last exponential, and integrate over k_2 using the condition $k_2.K'' = 0$. Make the change of variables $k_2^2 = p$ and integrate over k_2 (or p). Using Eq.(111) we find:

$$I_3(P,K') = \frac{1}{8}S_{d-m}S_m\Gamma(\frac{d-m}{2})\Gamma(\frac{m}{4})\int \frac{d^{d-m}q_1 d^m k_1}{q_1^2+(k_1^2)^2}$$
$$\int_0^{\infty}\int_0^{\infty} d\alpha_1 d\alpha_2 (\alpha_1+\alpha_2)^{-[\frac{(d-m)}{2}+\frac{m}{4}]} exp(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}[(q_1+P)^2+((k_1+K')^2)^2]). \tag{129}$$

The parametric integrals can be solved as before and we have:

$$I_3(P,K') = \frac{1}{8}S_{d-m}S_m\Gamma(\frac{d-m}{2})\Gamma(\frac{m}{4})\int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2+(k_1^2)^2)[(q_1+P)^2+((k_1+K')^2)^2]^{\frac{\epsilon_L}{2}}}. \tag{130}$$

We can now use Eq.(94) and absorbing the angular geometric factor for the first loop integral we obtain:

$$I_3(P,K') = \frac{1}{\epsilon_L}(1+[i_2]_m\epsilon_L)\int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2+(k_1^2)^2)[(q_1+P)^2+((k_1+K')^2)^2]^{\frac{\epsilon_L}{2}}}. \tag{131}$$

Let $i_3(P,K')$ be the last integral above. In the remainder, we employ a Feynman parameter. The integral $i_3(P,K')$ can be expressed in the following form:

$$i_3(P,K') = \frac{\Gamma(1+\frac{\epsilon_L}{2})}{\Gamma(\frac{\epsilon_L}{2})}\int_0^1 dx x^{\frac{\epsilon_L}{2}-1}$$
$$\int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2+2xP.q_1+xP^2+(1-x)(k_1^2)^2+x((k_1+K')^2)^2)^{1+\frac{\epsilon_L}{2}}}. \tag{132}$$

After that, take the orthogonality condition $k_1.K' = 0$. In order to solve the integral over the quadratic momenta q_1 we shall make use of the relation:

$$\int \frac{d^{d-m}q_1}{(q_1^2+2k.q_1+m^2)^{\alpha}} = \frac{1}{2}\frac{\Gamma(\frac{(d-m)}{2})\Gamma(\alpha-\frac{(d-m)}{2})(m^2-k^2)^{\frac{(d-m)}{2}-\alpha}}{\Gamma(\alpha)}S_{d-m}. \tag{133}$$

Thus, we obtain

$$i_3(P,K') = \frac{S_{d-m}}{2}\frac{\Gamma(2-\frac{m}{4}+\epsilon_L)\Gamma(-1+\frac{m}{4}+\epsilon_L)}{\Gamma(\frac{\epsilon_L}{2})}\int_0^1 dx x^{\frac{\epsilon_L}{2}-1}$$
$$\int \frac{d^m k_1}{((k_1^2)^2+P^2x(1-x)+x(2k_1^2K'^2+(K'^2)^2))^{-1+\frac{m}{4}+\epsilon_L}}. \tag{134}$$

Now, using Eq.(114), we can integrate over the quartic momenta k_1 obtaining:

$$i_3(P,K') = \frac{S_{d-m}S_m}{8}\frac{\Gamma(2-\frac{m}{4}+\frac{\epsilon_L}{2})\Gamma(\frac{m}{4})\Gamma(-1+\epsilon_L)}{\Gamma(\frac{\epsilon_L}{2})}$$
$$\times\int_0^1 dx x^{\frac{\epsilon_L}{2}-1}[x(1-x)(P^2+(K'^2)^2)]^{1-\epsilon_L}. \tag{135}$$

Expanding the resulting Γ functions and absorbing the geometric angular factor discussed above, we find:

$$I_3(P,K') = (P^2+(K'^2)^2)\frac{-1}{8\epsilon_L}(1+2[i_2]_m\epsilon_L - \frac{3}{4}\epsilon_L - 2\epsilon_L L_3(P,K')), \tag{136}$$

where

$$L_3(P,K') = \int_0^1 dx(1-x)ln[(P^2+(K'^2)^2)x(1-x)]. \tag{137}$$

At the symmetry points SP_1 or SP_2, it can be rewritten as

$$I_{3SP_1} = I_{3SP_2} = \frac{-1}{8\epsilon_L}(1+2[i_2]_m\epsilon_L + \frac{5}{4}\epsilon_L). \tag{138}$$

From the above equation we can derive the expressions:

$$I'_{3SP_1}(\equiv \frac{\partial I_{3SP_1}}{\partial P^2}) = I'_{3SP_2}(\equiv \frac{\partial I_{3SP_2}}{\partial K'^4}) = \frac{-1}{8\epsilon_L}(1+2[i_2]_m\epsilon_L + \frac{1}{4}\epsilon_L). \tag{139}$$

Let us now proceed to discuss the other required loop integrals. Consider

$$I_5 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^{d-m}q_3 d^m k_1 d^m k_2 d^m k_3}{(q_1^2+(k_1^2)^2)(q_2^2+(k_2^2)^2)(q_3^2+(k_3^2)^2)[(q_1+q_2-P)^2+((k_1+k_2-K')^2)^2]}$$
$$\times\frac{1}{(q_1+q_3-P)^2+((k_1+k_3-K')^2)^2}, \tag{140}$$

which is the three-loop diagram contributing to the two-point vertex function. This integral is symmetric under a change in the dummy loop momenta $q_2 \to q_3$ and $k_2 \to k_3$. Let us analyse the integrations over q_2, k_2 and q_3, k_3. We use the condition $k_2.(k_1-K')=0$ when integrating over k_2 as well as $k_3.(k_1-K')=0$ when performing the integral over k_3. The two internal bubbles, which are represented by the integrals over (q_2,k_2) and (q_3,k_3), respectively, give actually the same result, namely $I_2((q_1-P),(k_1-K'))$. Next take $P\to -P$, $K'\to -K'$. Therefore, we obtain the following expression:

$$I_5(P,K') = \frac{1}{\epsilon_L^2}(1+2[i_2]_m\epsilon_L)\int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2+(k_1^2)^2)[(q_1+P)^2+((k_1+K')^2)^2]^{\epsilon_L}}. \tag{141}$$

We employ a Feynman parameter in analogy to what was done in the calculation of I_3 and working out the details we find:

$$I_5(P,K') = (P^2+(K'^2)^2)\frac{-1}{6\epsilon_L^2}(1+3[i_2]_m\epsilon_L - \epsilon_L - 3\epsilon_L L_3(P,K')), \tag{142}$$

At the symmetry points SP_1, SP_2 we find:

$$I^{'}_{5SP_1}(\equiv \frac{\partial I_{5SP_1}}{\partial P^2}) = I^{'}_{5SP_2}(\equiv \frac{\partial I_{5SP_2}}{\partial K'^4}) = \frac{-1}{6\epsilon_L^2}(1 + 3[i_2]_m\epsilon_L + \frac{1}{2}\epsilon_L). \tag{143}$$

We are left with the task of calculating one of the two-loop diagrams contributing to the four-point function

$$\begin{aligned} I_4 \quad &= \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{(q_1^2 + (k_1^2)^2)\left((P-q_1)^2 + ((K'-k_1)^2)^2\right)(q_2^2 + (k_2^2)^2)} \\ &\qquad \times \frac{1}{(q_1 - q_2 + p_3)^2 + ((k_1 - k_2 + k_3')^2)^2} \quad . \end{aligned} \tag{144}$$

Notice that $P = p_1 + p_2$, p_i $(i = 1, ..., 3)$ are external momenta perpendicular to the competing axes whereas $K' = k_1' + k_2'$, and k_i' $(i = 1, ..., 3)$ are the external momenta along the competition directions. We can integrate first over the bubble (q_2, k_2). Using Schwinger parameters, and absorbing the geometric angular factor for the first bubble we obtain:

$$\begin{aligned} I_4 = \frac{1}{\epsilon_L}(1 + [i_2]_m\epsilon_L) &\int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2 + (k_1^2)^2)\left((P-q_1)^2 + ((K'-k_1)^2)^2\right)} \\ \times \ &\frac{1}{[(q_1 + p_3)^2 + ((k_1 + k_3')^2)^2]^{\frac{\epsilon_L}{2}}} \quad . \end{aligned} \tag{145}$$

Using a Feynman parameter one can write this in the form

$$\begin{aligned} I_4 = f_m(\epsilon_L) \int_0^1 dz &\int \frac{d^{d-m}q_1 d^m k_1}{[q_1^2 - 2z\, P.q_1 + zP^2 + (k_1^2)^2 + z((K'^2)^2 + 2K'^2 k_1^2)]^2} \\ \times &\frac{1}{[(q_1 + p_3)^2 + ((k_1 + k_3')^2)^2]^{\frac{\epsilon_L}{2}}} \ , \end{aligned} \tag{146}$$

where we defined the quantity $f_m(\epsilon_L) = \frac{1}{\epsilon_L}(1 + [i_2]_m\epsilon_L)$, which is the one-loop subdiagram with the angular factor already absorbed. Using another Feynman parameter to fold the two denominators in the last expression, integrating over p_1, k_1 (recalling the orthogonality condition already stated), the integral turns out to be

$$\begin{aligned} I_4 = \tfrac{1}{8} f_m(\epsilon_L) \, &\frac{\Gamma(\epsilon_L)\Gamma(\frac{m}{4})\Gamma(2 - \frac{m}{4} - \frac{\epsilon_L}{2})}{\Gamma\left(\frac{\epsilon_L}{2}\right)} \int_0^1 dy\, y\,(1-y)^{\frac{1}{2}\epsilon_L - 1} \int_0^1 dz \Big[yz(1-yz)(P^2 + (K'^2)^2) \\ &+ y(1-y)(p_3^2 + ((k_3')^2)^2) + 2yz(1-y)(p_3.P + (k_3')^2(K')^2) \Big]^{-\epsilon_L} S_{d-m}S_m. \end{aligned} \tag{147}$$

The integral over y is singular at $y = 1$ when $\epsilon_L = 0$. We only need to replace the value $y = 1$ inside the integral over z [26], and integrate over y afterwards, obtaining after the absorption of the geometric factor

$$I_4 = \frac{1}{2\epsilon_L^2}\Big(1 + 2\,[i_2]_m\epsilon_L - \frac{3}{2}\epsilon_L - \epsilon_L L(P, K')\Big). \tag{148}$$

This is the most appropriate form to carry out the renormalization using minimal subtraction. In terms of normalization conditions, we find the value of this integral at the symmetry points discussed before:

$$I_{4SP_1} = I_{4SP_2} = \frac{1}{2\epsilon_L^2}\Big(1 + 2\ [i_2]_m \epsilon_L + \frac{1}{2}\epsilon_L\Big). \tag{149}$$

Thus, we have successfully devised a new regularization procedure to calculate Feynman integrals whose propagators have quartic powers of momenta. It is tempting to define the measure of the m-dimensional sphere in terms of a half integer measure. In fact, taking $k = p^{2n}$, one has $d^m k \equiv d^{\frac{m}{2n}} p = \frac{1}{2n} p^{\frac{m}{2n}-1} dp d\Omega_m$. Hence, the approximation ammounts to take the new "measure" $d^{\frac{m}{2n}} p$ invariant under translations $p' = p + a$.

Note that this approximation is much better than the former dissipative approximation for the following reasons. First, we do not have to introduce any diagram factor, since after absorbing the geometric angular factor the leading singularities in ϵ_L are the same as those in the standard ϕ^4 field theory present in ϵ $(m = 0)$. Second, we have an expression in terms of arbitrary external momenta, which permits the computation of all the critical exponents in a completely independent manner using renormalization group transformations either perpendicular or parallel to the competition axes. Third, this can be easily adapted to the isotropic behavior. The three weak points of the dissipative approximation have been overcome in the orthogonal approximation.

B. Isotropic

The new orthogonal approximation can now be used to obtain the solutions to the Feynman integrals in the isotropic case. At the Lifshitz point $\delta_0 = 0$ all the quadratic momenta disappear and only quartic momenta are defined.

Consider the integral

$$I_2 = \int \frac{d^m k}{\left((k + K')^2\right)^2 (k^2)^2} \quad . \tag{150}$$

It is the isotropic counterpart of the one-loop integral contributing to the four-point function. Using two Schwinger parameters and the orthogonality condition $k.K' = 0$, we find:

$$I_2(K') = \int d^m k \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 e^{-(\alpha_1+\alpha_2)(k_2^2)^2} e^{-2\alpha_2 K'^2 k^2} e^{-\alpha_2 (K'^2)^2}. \tag{151}$$

Now performing the transformation $k^2 = p$ the volume element transforms to $d^m k = \frac{1}{2} p^{\frac{m}{2}-1} dp d\Omega_m \equiv d^{\frac{m}{2}} p$. The former quartic integral turns into a quadratic integral over p. After neglecting the infinite terms which change the measure $d^{\frac{m}{2}} k$ under the translation $y' = y + \frac{b}{2a}$, only the leading contribution is sorted out and we have

$$\int d^m k e^{-a(k^2)^2 - bk^2} = \int d^{\frac{m}{2}} p e^{-ap^2 - bp} \cong a^{-\frac{m}{4}} e^{\frac{b^2}{4a}} \frac{1}{4} \Gamma(\frac{m}{4}) S_m. \tag{152}$$

Replacing this result into the expression of $I_2(K')$, one finds

$$I_2(K') = \frac{1}{4}\Gamma(\frac{m}{4}) S_m \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 (\alpha_1 + \alpha_2)^{-\frac{m}{4}} \exp(-\frac{\alpha_1 \alpha_2 (K'^2)^2}{\alpha_1 + \alpha_2}). \tag{153}$$

Now, use a change of variables and a rescaling to realize the remaining parametric integrals analogously to what was done in the anisotropic case. Making the continuation $m = 8 - \epsilon_L$, the integral can be expressed in the following intermediate step:

$$I_2(K') = \frac{1}{4}\Gamma(2-\frac{\epsilon_L}{4})\Gamma(\frac{\epsilon_L}{4})S_m\int_0^1 dv[v(1-v)(K'^2)^2]^{-\frac{\epsilon_L}{4}}, \tag{154}$$

and the integration over v produces the result

$$I_2(K') = \frac{S_m}{\epsilon_L}[1-\frac{\epsilon_L}{4}(1+L(K'^2))]. \tag{155}$$

We absorb the factor of S_m in this integral through a redefinition of the coupling constant. Henceafter we shall absorb this factor when calculating each loop integral in analogy to the discussion for the anisotropic behavior. Note that this absorption factor is different from the one appearing for the anisotropic case in the limit $d \rightarrow m = 8-\epsilon_L$. In the anisotropic case the geometric angular factor becomes singular in the above isotropic limit indicating the failure of this attempt of extrapolating from one case to another. This is a more compelling technical reason for the statement that the isotropic and anisotropic cases are completely distinct. Therefore,

$$I_2(K') = \frac{1}{\epsilon_L}[1-\frac{\epsilon_L}{4}(1+L(K'^2))]. \tag{156}$$

In terms of a symmetry point $(K'^2)^2 = 1$ convenient whenever normalization conditions are used, we obtain

$$I_2(K') = \frac{1}{\epsilon_L}[1+\frac{\epsilon_L}{4}]. \tag{157}$$

We can go on to evaluate the other required higher loop integrals. The systematics is the same: solve the subdiagrams using the intermediate step Eq.(153) and then use Feynman parameters to solve the parametric integrals left over.

Let us calculate

$$I_3 = \int \frac{d^m k_1 d^m k_2}{((k_1+k_2+K')^2)^2 (k_1^2)^2 (k_2^2)^2} \quad , \tag{158}$$

the two-loop "sunset" Feynman diagram of the two-point function in the isotropic case. Integrate first over k_2. Take $K'' = k_1 + K'$ and use the condition $k_2.K'' = 0$ to obtain:

$$I_3 = \frac{1}{\epsilon_L}[1+\frac{\epsilon_L}{4}]\int \frac{d^m k_1}{[((k_1+K')^2)^2]^{\frac{\epsilon_L}{4}}(k_1^2)^2} \; . \tag{159}$$

Now using a Feynman parameter, integrating over k_1, taking $m = 8-\epsilon_L$, and employing the formula

$$\int \frac{d^{\frac{m}{2}}q}{(q^2+2k.q+m^2)^{\alpha}} \cong \frac{1}{4}\frac{\Gamma(\frac{m}{4})\Gamma(\alpha-\frac{m}{4})(m^2-k^2)^{\frac{m}{4}-\alpha}S_m}{\Gamma(\alpha)}, \tag{160}$$

one can express I_3 as

$$I_3 = \frac{S_m\Gamma(2-\frac{\epsilon_L}{4})\Gamma(-1+\frac{\epsilon_L}{2})}{4\epsilon_L\Gamma(\frac{\epsilon_L}{4})}[1+\frac{\epsilon_L}{4}]\int_0^1 dx[(K'^2)^2 x(1-x)]^{1-\frac{\epsilon_L}{2}}x^{-1+\frac{\epsilon_L}{4}}. \tag{161}$$

We can rewrite this expression in terms of the integral $L_3(K'^2)$ defined in the last section in the form

$$I_3 = \frac{S_m \Gamma(2-\frac{\epsilon_L}{4})\Gamma(-1+\frac{\epsilon_L}{2})}{4\epsilon_L \Gamma(\frac{\epsilon_L}{4})}[1+\frac{\epsilon_L}{4}][\frac{1}{2}-\frac{3\epsilon_L}{16}-\frac{\epsilon_L}{2}L_3(K'^2)]. \tag{162}$$

Expanding the Gamma functions and absorbing S_m, it is easy to show that

$$I_3 = -\frac{(K'^2)^2}{16\epsilon_L}[1+\epsilon_L(\frac{1}{8}-L_3(K'^2))]. \tag{163}$$

At the symmetry point, this can be expressed as:

$$I_3 = -\frac{1}{16\epsilon_L}[1+\frac{9}{8}\epsilon_L], \tag{164}$$

leading to

$$\frac{\partial I_3}{\partial (K'^2)^2}|_{SP_3} = I_3^{'} = -\frac{1}{16\epsilon_L}[1+\frac{5}{8}\epsilon_L]. \tag{165}$$

We can carry out the calculation of the other integrals using the same reasoning. The 3-loop integral I_5 is given by

$$I_5 = \int \frac{d^m k_1 d^m k_2 d^m k_3}{((k_1+k_2+K')^2)^2((k_1+k_3+K')^2)^2(k_1^2)^2(k_2^2)^2(k_3^2)^2} \quad , \tag{166}$$

where we took for convenience $K' \to -K'$. The integrals over k_2 and k_3 are identical. Hence

$$I_5 = \frac{1}{\epsilon_L^2}[1+\frac{\epsilon_L}{2}] \int \frac{d^m k_1}{[((k_1+K')^2)^2]^{\frac{\epsilon_L}{2}}(k_1^2)^2} \ . \tag{167}$$

Employing the same techniques as in the calculation of I_3 we obtain

$$I_5 = -\frac{(K'^2)^2}{12\epsilon_L^2}[1+\epsilon_L(\frac{1}{4}-\frac{3}{2}L_3(K'^2))]. \tag{168}$$

At the symmetry point, the derivative of I_5 with respect to the external momenta is given by

$$\frac{\partial I_5}{\partial (K'^2)^2}|_{SP_3} = I_5^{'} = -\frac{1}{12\epsilon_L^2}[1+\epsilon_L]. \tag{169}$$

The two-loop graph contributing to the 4-point function in the isotropic situation is

$$I_4 \ = \int \frac{d^m k_1 d^m k_2}{(k_1^2)^2((K'-k_1)^2)^2 (k_2^2)((k_1-k_2+k_3')^2)^2} \ , \tag{170}$$

where $K' = k_1' + k_2'$. It can be integrated using the orthogonal approximation following the same steps of the calculation of the anisotropic counterpart. We simply quote the result

$$I_4(K'^2) = \frac{1}{2\epsilon_L^2}[1 - \frac{\epsilon_L}{4}(1 + 2L(K'^2))]. \tag{171}$$

At the symmetry point it is given by:

$$I_4(K'^2) = \frac{1}{2\epsilon_L^2}[1 + \frac{3\epsilon_L}{4}]. \tag{172}$$

It is worth to point out that the integrals I_3 and I_5 have not the same leading singularities as in the usual ϕ^4. Therefore, any attempt to use the counterterms of the usual ϕ^4 theory would lead to erroneous results for the critical exponents in the isotropic case.

A recent calculation of the critical exponents η_{L4} and ν_{L4} for the isotropic case was presented by Diehl and Shpot who fixed all the leading singularities equal to those appearing in the standard ϕ^4 theory [30]. Moreover they "choose" the following angular factor:

$$F_d = 2^{1-d}\pi^{-\frac{d}{2}}\frac{\Gamma(5-\frac{d}{2})\Gamma^2(\frac{d}{2}-2)}{\Gamma(d-4)}. \tag{173}$$

If one sets $d = 8 - \epsilon_L$, whenever the Feynman integral under consideration presents a double pole in ϵ_L, this angular factor will give contribution to the simple poles in ϵ_L. This happens for example in the two-loop integrals I_2^2 and I_4. Then, their calculation of these critical exponents cannot be trusted, since further ϵ_L^{-1} terms were not taken into account in the evaluation of those integrals. It seems that this factor was used to reproduce the original critical exponents η_{L4} and ν_{L4} from the seminal paper [1].

Here the geometric angular factor is determined simply by requiring that only I_2 has the same leading singularity as in the standard ϕ^4 theory. In that case, it is simply given by the area of the m-dimensional sphere.

We can now discuss the fixed points and critical exponents for arbitrary m-axial behavior. Note that in the isotropic case, only I_2 and I_4 have the same leading singularities as in the standard ϕ^4 field theory. This will lead to a different fixed point for the isotropic behavior, as it is going to be shown in a moment.

VI. THE CRITICAL EXPONENTS FOR THE ANISOTROPIC BEHAVIORS

A. The dissipative approximation

The critical exponents were first calculated using this approximation for the uniaxial case and the generalization was soon presented for the m-axial case. Details can be found in Refs. [20, 23]. Here we shall simply quote the results.

The fixed point at two-loop order is given by:

$$u^* = \frac{6}{8+N}\,\epsilon_L\left\{1 + \epsilon_L\left[\left(\frac{4(5N+22)}{(8+N)^2} - 1\right)[i_2]_m - \frac{(2+N)}{(8+N)^2}\right]\right\} . \tag{174}$$

It can be used to obtain the critical exponents η_{L2} and ν_{L2}:

$$
\eta_{L2} = \frac{1}{2}\epsilon_L^2 \frac{2+N}{(8+N)^2}
+ \epsilon_L^3 \frac{(2+N)}{(8+N)^2}\left[\left(\frac{4(5N+22)}{(8+N)^2} - \frac{1}{2}\right)[i_2]_m + \frac{1}{8-m} - \frac{2+N}{(8+N)^2}\right] , \tag{175}
$$

$$
\nu_{L2} = \frac{1}{2} + \frac{1}{4}\epsilon_L \frac{2+N}{8+N}
+ \frac{1}{8}\frac{(2+N)}{(8+N)^3}\left[2(14N+40)\,[i_2]_m - 2(2+N) + (8+N)(3+N)\right]\epsilon_L^2 . \tag{176}
$$

Using the scaling law $\gamma_L = \nu_{L2}(2-\eta_{L2})$, the exponent γ_L is

$$
\gamma_L = 1 + \frac{1}{2}\epsilon_L \frac{2+N}{8+N}
+ \frac{1}{4}\frac{(2+N)}{(8+N)^3}\left[12 + 8N + N^2 + 4\,[i_2]_m\,(20+7N)\right]\epsilon_L^2 . \tag{177}
$$

The evaluation of Feynman diagrams used to obtain these results have some inconveniences as discussed before: the introduction of diagram factors for the integrals I_3' and I_5' (which are divergent when $m = 8$) is the main trouble.

The RG analysis by Mergulhão and Carneiro has been used in a attempt to extend the calculation for all m. Using the fact that the quartic momenta scale including σ and the quadratic external momenta scales are equal Diehl and Shpot considered the anisotropic problem for general m [31, 32]. In their first work [31], they worked directly in position space. After that, using a hybrid approach, going to coordinate or momentum space using the free propagator (scaling function) in coordinate space to make the transition according to the necessity, they calculated the critical exponents using a minimal subtraction procedure which sets the external quartic momenta scales to zero. However, there is a small discrepancy among their results for the critical exponents in the cases $m = 2, 6$ when compared with Mergulhão and Carneiro's results using normalization conditions [32]. For the anisotropic cases $m \neq d$, the exponents are given in terms of integrals to be performed numerically. These numerical integrals are meaningful solely if one separates the integration limits on the variable $\mathbf{v} = \sigma_0 \mathbf{x}_{\parallel} \mathbf{x}_{\perp}$ using the scaling and related functions in the coordinate space representation in the integrand up to the maximum value of $|\mathbf{v}|$ at $|v_0| = 9.3$, and replacing the asymptotic value of these functions for greater values of $\mathbf{v}$. Note that as the quartic and quadratic external momenta are not independent, σ *cannot* be taken dimensionless as done by these authors following the invalid argument by Mergulhão and Carneiro. Thus, they erroneously concluded that the isotropic case could be encompassed in their expressions for the critical exponents in the limit $d \to m$ close to 8.

Furthermore, this alternative semianalytical method has some drawbacks. First, setting the quadratic momenta scale to zero makes impossible the transition from the anisotropic to the isotropic case, since the quadratic momenta scale is absent in this case and renormalization group transformations are defined only through the variation of the quartic momenta scale. Second, unfortunately the expression of the critical exponents for the anisotropic case

are rather cumbersome, given in terms of integrals to be performed numerically. Clearly the most convenient answer should present analytical coefficients for each order in ϵ_L, in analogy to the usual standard ϕ^4 theory describing the Ising model.

The calculation of the critical exponents using the dissipative approximation presented here has been criticized by Diehl and Shpot [33] because of the constraint introduced in the quartic loop momenta in higher loop Feynman integrals. Despite of this criticism, this approximation is in good agreement with recent high-precision numerical data based in Monte Carlo simulations for the ANNNI model [34, 22].

B. The orthogonal approximation

We turn now our attention to the most general approximation for calculating all the critical exponents. In order to check the results, we must calculate the critical exponents using two different renormalization schemes, namely the normalization conditions and minimal subtraction of dimensional poles. It is going to be shown now that the critical indices are the same irrespective of the use of either renormalization prescription.

1. Normalization conditions and critical exponents

We defined the bare coupling constants and renormalization functions as

$$u_{o\tau} = u_\tau(1 + a_{1\tau}u_\tau + a_{2\tau}u_\tau^2), \tag{178a}$$
$$Z_{\phi(\tau)} = 1 + b_{2\tau}u_\tau^2 + b_{3\tau}u_\tau^3, \tag{178b}$$
$$\bar{Z}_{\phi^2(\tau)} = 1 + c_{1\tau}u_\tau + c_{2\tau}u_\tau^2, \tag{178c}$$

where the constants $a_{i\tau}, b_{i\tau}, c_{i\tau}$ depend on Feynman integrals calculated at the convenient symmetry points. Depending on the symmetry point, we can calculate critical exponents corresponding to correlations perpendicular or parallel to the competing m-dimensional subspace.

The beta-functions and renormalization constants can be rewritten in terms of the constants defined above in the following form:

$$\beta_\tau = -\tau\epsilon_L u_\tau[1 - a_{1\tau}u_\tau + 2(a_{1\tau}^2 - a_{2\tau})u_\tau^2], \tag{179a}$$
$$\gamma_{\phi(\tau)} = -\tau\epsilon_L u_\tau[2b_{2\tau}u_\tau + (3b_{3\tau} - 2b_{2\tau}a_{1\tau})u_\tau^2], \tag{179b}$$
$$\bar{\gamma}_{\phi^2(\tau)} = \tau\epsilon_L u_\tau[c_{1\tau} + (2c_{2\tau} - c_{1\tau}^2 - a_{1\tau}c_{1\tau})u_\tau]. \tag{179c}$$

It is easy to obtain the coefficients above as functions of the integrals calculated at the symmetry points. They are given by

$$a_{1\tau} = \frac{N+8}{6\epsilon_L}[1 + [i_2]_m\epsilon_L], \tag{180a}$$
$$a_{2\tau} = (\frac{N+8}{6\epsilon_L})^2 + [\frac{(N+8)^2}{18}[i_2]_m - \frac{(3N+14)}{24}]\frac{1}{\epsilon_L}, \tag{180b}$$
$$b_{2\tau} = -\frac{(N+2)}{144\epsilon_L}[1 + (2[i_2]_m + \frac{1}{4})\epsilon_L], \tag{180c}$$

$$b_{3\tau} = -\frac{(N+2)(N+8)}{1296\epsilon_L^2} + \frac{(N+2)(N+8)}{108\epsilon_L}(-\frac{1}{4}[i_2]_m + \frac{1}{48}), \tag{180d}$$

$$c_{1\tau} = \frac{(N+2)}{6\epsilon_L}[1 + [i_2]_m\epsilon_L], \tag{180e}$$

$$c_{2\tau} = \frac{(N+2)(N+5)}{36\epsilon_L^2} + \frac{(N+2)}{3\epsilon_L}[\frac{(N+5)}{3}[i_2]_m - \frac{1}{4}]. \tag{180f}$$

This is enough to obtain the fixed points at $O(\epsilon_L^2)$. They are defined by $\beta_\tau(u_\tau^*) = 0$. All the integrals calculated at the symmetry points SP_1 and SP_2 look the same. The factor of $\tau = 1,2$ drops out at the fixed points, implying that the renormalization group transformations performed either over κ_1 or κ_2 will flow to the same fixed point given by $(u_1^* = u_2^* \equiv u^*)$

$$u^* = \frac{6}{8+N}\,\epsilon_L\Bigg\{1 + \epsilon_L\left[-[i_2]_m + \frac{(9N+42)}{(8+N)^2}\right]\Bigg\}\ . \tag{181}$$

The surprising feature of this approximation is that the critical exponents do not depend on $[i_2]_m$, as the ones obtained using the dissipative approximation. The functions $\gamma_{\phi(1)}$ and $\bar{\gamma}_{\phi^2(1)}$ can be written as

$$\gamma_{\phi(1)} = \frac{(N+2)}{72}[1 + (2[i_2]_m + \frac{1}{4})\epsilon_L]u_1^2 - \frac{(N+2)(N+8)}{864}u_1^3, \tag{182}$$

$$\bar{\gamma}_{\phi^2(1)} = \frac{(N+2)}{6}u_1[1 + [i_2]_m\epsilon_L - \frac{1}{2}u_1]. \tag{183}$$

Replacing the value of the fixed point inside these equations, using the relation among these functions and the critical exponents η_{L2} and ν_{L2}, we find:

$$\eta_{L2} = \frac{1}{2}\epsilon_L^2\,\frac{N+2}{(N+8)^2}[1 + \epsilon_L(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4})], \tag{184}$$

$$\nu_{L2} = \frac{1}{2} + \frac{(N+2)}{4(N+8)}\epsilon_L + \frac{1}{8}\frac{(N+2)(N^2+23N+60)}{(N+8)^3}\epsilon_L^2. \tag{185}$$

Notice that the coefficient of each power of ϵ_L is the same as in the pure ϕ^4 describing the Ising-like behavior. The reduction to the $m = 0$ case is even simpler using this approximation than the reduction to the $m = 0$ case using the dissipative approximation. Since the functions $\gamma_{\phi(2)} = 2\gamma_{\phi(1)}$ and $\bar{\gamma}_{\phi^2(2)} = 2\bar{\gamma}_{\phi^2(1)}$ as a consequence of $\beta_2 = 2\beta_1$, we immediately conclude that

$$\eta_{L4} = \epsilon_L^2\,\frac{(N+2)}{(N+8)^2}[1 + \epsilon_L(\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4})], \tag{186}$$

$$\nu_{L4} = \frac{1}{4} + \frac{(N+2)}{8(N+8)}\epsilon_L + \frac{1}{16}\frac{(N+2)(N^2+23N+60)}{(N+8)^3}\epsilon_L^2. \tag{187}$$

Thus, at $O(\epsilon_L^3)$, the relation $\eta_{L4} = 2\eta_{L2}$ is valid. At $O(\epsilon_L^2)$, the relation $\nu_{L4} = \frac{1}{2}\nu_{L2}$ is fulfilled. Thus the strong anisotropic scale invariance [24] is *exact* to the perturbative order considered here. The other exponents can be read from the scaling relations. As discussed before, they are ($\alpha_{L2} = \alpha_{L4} = \alpha_L$, etc.):

$$\gamma_L = 1 + \frac{(N+2)}{2(N+8)}\epsilon_L + \frac{(N+2)(N^2+22N+52)}{4(N+8)^3}\epsilon_L^2, \tag{188}$$

$$\alpha_L = \frac{(4-N)}{2(N+8)}\epsilon_L - \frac{(N+2)(N^2+30N+56)}{4(N+8)^3}\epsilon_L^2, \tag{189}$$

$$\beta_L = \frac{1}{2} - \frac{3}{2(N+8)}\epsilon_L + \frac{(N+2)(2N+1)}{2(N+8)^3}\epsilon_L^2, \tag{190}$$

$$\delta_L = 3 + \epsilon_L + \frac{(N^2+14N+60)}{2(N+8)^2}\epsilon_L^2. \tag{191}$$

Note that all these exponents reduce to the Ising-like case when $m = 0$. In order to check the correctness of these exponents, it is convenient to calculate them in another renormalization procedure, as we shall see next.

2. Minimal subtraction and critical exponents

Usually, in the minimal subtraction renormalization scale, one can have more than one coupling, but just one momenta scale, called in most textbooks μ [36] and named κ here. The dimensional redefinition performed for the quartic external momenta, allows the description of the anisotropic case with two independent momenta scales. The coupling constant has two independent flows, induced by κ_1 and κ_2.

If we want to calculate the critical exponents along the competition axes, we set the quadratic external momenta perpendicular to the competing subspace to zero. Thus, one can introduce the quartic momenta scale κ_2 in order to compute the normalization functions for arbitrary quartic external momenta and demanding that the dimensional poles (logarithmic divergences in the momenta) be minimally subtracted. On the other hand, the calculation of critical exponents perpendicular to the competing axes can be performed by setting the quartic external momenta to zero, introducing κ_1, calculating the normalization functions for arbitrary quadratic external momenta and requiring minimal subtraction.

In this section, we are not going to calculate explicitly the critical exponents. Instead, we are going to calculate the fixed point as well as the functions $\gamma_{\phi(\tau)}$ and $\bar{\gamma}_{\phi^2(\tau)}$ at the fixed point. As these functions at the fixed point are universal, they should be equal to the ones obtained using normalization conditions, leading to the same exponents in either renormalization scheme.

The dimensionless bare couplings and the renormalization functions are defined in minimal subtraction by

$$u_{0\tau} = u_\tau[1 + \sum_{i=1}^{\infty} a_{i\tau}(\epsilon_L)u_\tau^i], \tag{192a}$$

$$Z_{\phi(\tau)} = 1 + \sum_{i=1}^{\infty} b_{i\tau}(\epsilon_L)u_\tau^i, \tag{192b}$$

$$\bar{Z}_{\phi^2(\tau)} = 1 + \sum_{i=1}^{\infty} c_{i\tau}(\epsilon_L)u_\tau^i. \tag{192c}$$

The renormalized vertices

$$\Gamma^{(2)}_{R(\tau)}(k_\tau, u_\tau, \kappa_\tau) = Z_{\phi(\tau)}\Gamma^{(2)}_{(\tau)}(k_\tau, u_{0\tau}, \kappa_\tau), \tag{193a}$$

$$\Gamma^{(4)}_{R(\tau)}(k_{i\tau}, u_\tau, \kappa_\tau) = Z^2_{\phi(\tau)}\Gamma^{(4)}_{(\tau)}(k_{i\tau}, u_{0\tau}, \kappa_\tau), \tag{193b}$$

$$\Gamma^{(2,1)}_{R(\tau)}(k_{1\tau}, k_{2\tau}, p_\tau; u_\tau, \kappa_\tau) = \bar{Z}_{\phi^2(\tau)}\Gamma^{(2,1)}_{(\tau)}(k_{1\tau}, k_{2\tau}, p_\tau, u_{0\tau}, \kappa_\tau), \tag{193c}$$

are finite when $\epsilon_L \to 0$, order by order in u_τ. Note that the external momenta into the bare vertices are mutiplied by κ_τ^{-1}. Recall that $k_{i1} = p_i$ are the external momenta perpendicular to the competing axes, whereas $k_{i2} = k'_i$ are the external momenta parallel to the m-dimensional subspace. The coefficients $a_{i\tau}(\epsilon_L), b_{i\tau}(\epsilon_L)$ and $c_{i\tau}(\epsilon_L)$ are obtained by requiring that the poles in ϵ_L be minimally subtracted. The bare vertices can now be expressed as

$$\Gamma^{(2)}_{(\tau)}(k_\tau, u_{0\tau}, \kappa_\tau) = k_\tau^{2\tau}(1 - B_{2\tau}u_{0\tau}^2 + B_{3\tau}u_{0\tau}^3), \tag{194a}$$

$$\Gamma^{(4)}_{(\tau)}(k_{i\tau}, u_{0\tau}, \kappa_\tau) = \kappa_\tau^{\tau\epsilon}u_{0\tau}[1 - A_{2\tau}u_{0\tau} + (A^{(1)}_{2\tau} + A^{(2)}_{2\tau})u_{0\tau}^2], \tag{194b}$$

$$\Gamma^{(2,1)}_{(\tau)}(k_{1\tau}, k_{2\tau}, p_\tau; u_{0\tau}, \kappa_\tau) = 1 - C_{1\tau}u_{0\tau} + (C^{(1)}_{2\tau} + C^{(2)}_{2\tau})u_{0\tau}^2. \tag{194c}$$

Notice that $B_{2\tau}$ is proportional to the integral I_3 and $B_{3\tau}$ is proportional to I_5. Note that if $\tau = 1$, all the integrals should be replaced by their values at zero quartic external momenta. In case $\tau = 2$, those integrals are calculated at zero quadratic external momenta.

Explicitly, the coefficients are given by the following integrals:

$$A_{1\tau} = \frac{(N+8)}{18}[I_2(\frac{k_{1\tau}+k_{2\tau}}{\kappa_\tau}) + I_2(\frac{k_{1\tau}+k_{3\tau}}{\kappa_\tau}) + I_2(\frac{k_{2\tau}+k_{3\tau}}{\kappa_\tau})], \tag{195a}$$

$$A^{(1)}_{2\tau} = \frac{(N^2+6N+20)}{108}[I_2^2(\frac{k_{1\tau}+k_{2\tau}}{\kappa_\tau}) + I_2^2(\frac{k_{1\tau}+k_{3\tau}}{\kappa_\tau}) + I_2^2(\frac{k_{2\tau}+k_{3\tau}}{\kappa_\tau})], \tag{195b}$$

$$A^{(2)}_{2\tau} = \frac{(5N+22)}{54}[I_4(\frac{k_{i\tau}}{\kappa_\tau}) + 5\ permutations], \tag{195c}$$

$$B_{2\tau} = \frac{(N+2)}{18}I_3(\frac{k_\tau}{\kappa_\tau}), \tag{195d}$$

$$B_{3\tau} = \frac{(N+2)(N+8)}{108}I_5(\frac{k_\tau}{\kappa_\tau}), \tag{195e}$$

$$C_{1\tau} = \frac{N+2}{18}[I_2(\frac{k_{1\tau}+k_{2\tau}}{\kappa_\tau}) + I_2(\frac{k_{1\tau}+k_{3\tau}}{\kappa_\tau}) + I_2(\frac{k_{2\tau}+k_{3\tau}}{\kappa_\tau})], \tag{195f}$$

$$C^{(1)}_{2\tau} = \frac{(N+2)^2}{108}[I_2^2(\frac{k_{1\tau}+k_{2\tau}}{\kappa_\tau}) + I_2^2(\frac{k_{1\tau}+k_{3\tau}}{\kappa_\tau}) + I_2^2(\frac{k_{2\tau}+k_{3\tau}}{\kappa_\tau})], \tag{195g}$$

$$C^{(2)}_{2\tau} = \frac{N+2}{36}[I_4(\frac{k_{i\tau}}{\kappa_\tau}) + 5\ permutations]. \tag{195h}$$

This is sufficient to determine the normalization constants to the loop order desired. Requiring minimal subtraction of dimensional poles for the renormalized vertex parts quoted above, all the logarithmic integrals in the external momenta appearing in I_2, I_3, I_4, and I_5 cancell each other. The result is that the normalization functions and coupling constants can be expressed in the form:

$$u_{0\tau} = u_\tau(1 + \frac{(N+8)}{6\epsilon_L}u_\tau + [\frac{(N+8)^2}{36\epsilon_L^2} - \frac{(3N+14)}{24\epsilon_L}]u_\tau^2), \tag{196a}$$

$$Z_{\phi(\tau)} = 1 - \frac{N+2}{144\epsilon_L}u_\tau^2 + [-\frac{(N+2)(N+8)}{1296\epsilon_L^2} + \frac{(N+2)(N+8)}{5184\epsilon_L}]u_\tau^3, \tag{196b}$$

$$\bar{Z}_{\phi^2(\tau)} = 1 + \frac{N+2}{6\epsilon_L}u_\tau + [\frac{(N+2)(N+5)}{36\epsilon_L^2} - \frac{(N+2)}{24\epsilon_L}]u_\tau^2). \tag{196c}$$

From the renormalization functions one can obtain:

$$\gamma_{\phi(\tau)} = \tau[\frac{(N+2)}{72}u_\tau^2 - \frac{(N+2)(N+8)}{1728}u_\tau^3], \tag{197}$$

$$\bar{\gamma}_{\phi^2(\tau)} = \tau\frac{(N+2)}{6}u_\tau[1 - \frac{1}{2}u_\tau]. \tag{198}$$

The fixed points are defined by $\beta_\tau(u_\tau^*) = 0$. Then, it is found that the fixed points generated by renormalization group transformations over either κ_1 or κ_2 are the same and is given by:

$$u_\tau^* = \frac{6}{8+N}\epsilon_L\left\{1 + \epsilon_L\left[\frac{(9N+42)}{(8+N)^2}\right]\right\} . \tag{199}$$

Substitution of this result into the renormalization constants will give at the fixed point $\gamma_{\phi(\tau)}^* = \eta_\tau$, where η_τ are given by Eqs. (184) and (186). In addition, we have

$$\bar{\gamma^*}_{\phi^2(\tau)} = \tau\frac{(N+2)}{(N+8)}\epsilon_L[1 + \frac{6(N+3)}{(N+8)^2}\epsilon_L]. \tag{200}$$

This leads to the same exponents ν_τ given in Eqs. (185) and (187), obtained there via normalization conditions. Therefore, we have proven the consistency of this picture for the anisotropic Lifshitz critical behavior, since the critical indices are independent of the renormalization procedure.

3. Discussion

The exponent η_{L2} obtained here agrees with the calculation performed independently by Mukamel [37]. Nevertheless, the exponent η_{L4} presented here is at variance with Mukamel's [37] and, therefore, with the result obtained by Hornreich and Bruce [38] since both works agree with each other.

We are now in position to compare our results with those obtained for the ANNNI model in three-dimensional space ($\epsilon_L = 1.5$) representing the uniaxial ($m = 1$) case using Monte Carlo simulations [34]. From the numerical viewpoint, there is no sensitive difference among the results presented either using the dissipative approximation or the orthogonal approximation for the critical exponents perpendicular to the competition axes. Within the two significative algarisms precision the exponents using either approximation are given by $\eta_{L2} = 0.04$ and $\nu_{L2} = 0.73$.

The deviations start in the calculation of $\gamma_{L2} = \gamma_L$. In the dissipative approximation the ϵ_L-expansion yields $\gamma_L = 1.45$. A numerical interpretation has been proposed recently in order to improve the results obtained via the ϵ_L-expansion when the perturbative parameter ϵ_L is greater than 1 [22]. There it was argued that the neglected $O(\epsilon_L^3)$ could be relevant to

the calculation of, say, γ_L. The basic idea is to replace the numerical values of ν_{L2} and η_{L2} directly into the scaling laws in order to obtain the other critical exponents. In this way one obtains $\gamma_L = 1.43$, $\alpha_L = 0.18$ and $\beta_L = 0.20$.

On the other hand, using the ϵ_L-expansion results for γ_L, α_L and β_L obtained via the orthogonal approximation one finds $\gamma_L = 1.42$, $\alpha_L = 0.05$ and $\beta_L = 0.26$. The numerical "ansatz" described above gives again $\gamma_L = 1.43$, $\alpha_L = 0.18$ and $\beta_L = 0.20$, since η_{L2} and ν_{L2} have the same numerical values in either approximation. These numbers should be compared with the newest Monte Carlo simulations output, namely, $\gamma_L = 1.36 \pm 0.03$, $\alpha_L = 0.18 \pm 0.02$ and $\beta_L = 0.238 \pm 0.005$.

Thus, the greater the mean field values for the exponents, the better are the their numerical values using the ϵ_L-expansion when ϵ_L is not a small number. Otherwise, the numerical "ansatz" yields a rather good agreement with the Monte Carlo results, as in the case for the exponents α_L and β_L. This shows that the new results displayed here are consistent with the best numerical values available for the ANNNI model.

VII. THE CRITICAL EXPONENTS FOR THE ISOTROPIC SYSTEMS

As the isotropic behavior presents just one external momenta scale, its analysis is simpler than the one used to describe the anisotropic behavior, where two external momenta scales are present. Besides, the only manner to attack this problem is to use the orthogonal approximation, for the dissipative approximation does not work as it was discussed before.

A. Critical exponents in normalization conditions

The bare coupling constants and renormalization functions are defined as

$$u_{03} = u_3(1 + a_{13}u_3 + a_{23}u_3^2), \tag{201a}$$
$$Z_{\phi(3)} = 1 + b_{23}u_3^2 + b_{33}u_3^3, \tag{201b}$$
$$\bar{Z}_{\phi^2(3)} = 1 + c_{13}u_3 + c_{23}u_3^2, \tag{201c}$$

where the constants a_{i3}, b_{i3}, c_{i3} depend on Feynman integrals calculated at the symmetry point named henceafter SP_3. Only the external momenta scale κ_3 parallel to the competing m-dimensional subspace arises in this isotropic case.

The beta-function and renormalization constants are written in terms of the constants defined above in the following manner:

$$\beta_3 = -\epsilon_L u_3[1 - a_{13}u_3 + 2(a_{13}^2 - a_{23})u_3^2], \tag{202a}$$
$$\gamma_{\phi(3)} = -\epsilon_L u_3[2b_{23}u_3 + (3b_{33} - 2b_{23}a_{13})u_3^2], \tag{202b}$$
$$\bar{\gamma}_{\phi^2(3)} = \epsilon_L u_3[c_{13} + (2c_{23} - c_{13}^2 - a_{13}c_{13})u_3]. \tag{202c}$$

The coefficients above are obtained as functions of the integrals calculated at the symmetry point. They read

$$a_{13} = \frac{N+8}{6\epsilon_L}[1 + \frac{1}{4}\epsilon_L], \tag{203a}$$

$$a_{23} = (\frac{N+8}{6\epsilon_L})^2 + [\frac{2N^2+23N+86}{144\epsilon_L}], \tag{203b}$$

$$b_{23} = -\frac{(N+2)}{288\epsilon_L}[1 + \frac{5}{8}\epsilon_L], \tag{203c}$$

$$b_{33} = -\frac{(N+2)(N+8)}{2592\epsilon_L^2} - \frac{(N+2)(N+8)}{20736\epsilon_L}, \tag{203d}$$

$$c_{13} = \frac{(N+2)}{6\epsilon_L}[1 + \frac{1}{4}\epsilon_L], \tag{203e}$$

$$c_{23} = \frac{(N+2)(N+5)}{36\epsilon_L^2} + \frac{(N+2)(2N+7)}{144\epsilon_L}. \tag{203f}$$

The fixed point is defined by $\beta_3(u_3^*) = 0$. Therefore, it is given by

$$u_3^* = \frac{6}{8+N}\,\epsilon_L\left\{1 + \epsilon_L\frac{1}{2}\left[-\frac{1}{2} + \frac{(9N+42)}{(8+N)^2}\right]\right\}\ . \tag{204}$$

Note that this fixed point is different from that appearing in the anisotropic behavior and cannot be obtained from it in a smooth way. The functions $\gamma_{\phi(3)}$ and $\bar{\gamma}_{\phi^2(3)}$ can be written as

$$\gamma_{\phi(3)} = \frac{(N+2)}{144}[1 + \frac{5}{8}\epsilon_L]u_3^2 - \frac{(N+2)(N+8)}{3456}u_3^3, \tag{205}$$

$$\bar{\gamma}_{\phi^2(3)} = \frac{(N+2)}{6}u_3[1 + \frac{1}{4}\epsilon_L - \frac{1}{4}u_3]. \tag{206}$$

Replacing the value of the fixed point inside these equations, using the relation among these functions and the critical exponents η_{L4} and ν_{L4}, we find:

$$\eta_{L4} = \frac{1}{4}\epsilon_L^2\,\frac{N+2}{(N+8)^2}[1 + \epsilon_L(\frac{3(3N+14)}{(N+8)^2} - \frac{1}{8})], \tag{207}$$

$$\nu_{L4} = \frac{1}{4} + \frac{(N+2)}{16(N+8)}\epsilon_L + \frac{1}{256}\frac{(N+2)(N^2+23N+60)}{(N+8)^3}\epsilon_L^2. \tag{208}$$

These exponents are different from those originally obtained in Ref. [1]. The coefficient of the ϵ_L^2 term in the exponent η_{L4} is positive, consistent with its counterpart in the anisotropic cases as well as in the Ising-like case. One learns that only the quartic momenta is not sufficient to induce its change of sign. The exponent ν_{L4} agrees at $O(\epsilon_L)$ with that presented in Ref. [1] but naturally disagrees at $O(\epsilon_L^2)$, since it depends on the value of η_{L4} at $O(\epsilon_L^2)$. Besides, the critical index η_{L4} is obtained at $O(\epsilon_L^3)$ here for the first time.

Now using the scaling relations derived for the isotropic case we obtain immediately

$$\gamma_{L4} = 1 + \frac{(N+2)}{4(N+8)}\epsilon_L + \frac{(N+2)(N^2+19N+28)}{64(N+8)^3}\epsilon_L^2, \tag{209}$$

$$\alpha_{L4} = \frac{(4-N)}{4(N+8)}\epsilon_L + \frac{(N+2)(N^2+9N+68)}{32(N+8)^3}\epsilon_L^2, \tag{210}$$

$$\beta_{L4} = \frac{1}{2} - \frac{3}{4(N+8)}\epsilon_L - \frac{(N+2)(N^2+N+108)}{64(N+8)^3}\epsilon_L^2, \tag{211}$$

$$\delta_{L4} = 3 + \frac{1}{2}\epsilon_L + \frac{(N^2+14N+60)}{8(N+8)^2}\epsilon_L^2. \tag{212}$$

These exponents are obtained here for the first time at $O(\epsilon_L^2)$. Formerly the lack of a set of scaling laws for the isotropic case did not allow these findings. In order to check these results, let us analyse the situation using the minimal subtraction scheme.

B. Critical exponents in minimal subtraction

We proceed analogously as in the isotropic case. We just replace the subscript $\tau = 3$ and keep in mind that the Feynman integrals are calculated in the isotropic case $d = m$ close to 8. Minimal subtraction of dimensional poles in the renormalized vertex $\Gamma^{(4)}_{R(3)}$ implies that the bare dimensionless coupling constant can be expressed in the form:

$$u_{03} = u_3[1 + \frac{(N+8)}{6\epsilon_L}u_3 + (\frac{(N+8)^2}{36\epsilon_L^2} - \frac{(3N+14)}{48\epsilon_L})u_3^2]. \tag{213}$$

The fixed point can be easily found to be

$$u_3^* = \frac{6}{(N+8)}\epsilon_L + \frac{9(3N+14)}{(N+8)^3}\epsilon_L^2. \tag{214}$$

The normalization constants are given by:

$$\begin{aligned} Z_{\phi(3)} &= 1 - \frac{(N+2)}{288\epsilon_L}u_3^2 \\ &+[-\frac{(N+2)(N+8)}{2592\epsilon_L^2} + \frac{(N+2)(N+8)}{20736\epsilon_L}]u_3^3, \end{aligned} \tag{215}$$

$$\begin{aligned} \bar{Z}_{\phi^2(3)} &= 1 + \frac{(N+2)}{6\epsilon_L}u_3 \\ &+[\frac{(N+2)(N+5)}{36\epsilon_L^2} - \frac{(N+2)}{48\epsilon_L}]u_3^2. \end{aligned} \tag{216}$$

The functions $\gamma_{\phi(3)}$ and $\bar{\gamma}_{\phi^2(3)}$ are given by the following expressions:

$$\gamma_{\phi(3)} = \frac{(N+2)}{144}u_3^2 - \frac{(N+2)(N+8)}{6912}u_3^3, \tag{217a}$$

$$\gamma_{\phi^2(3)} = \frac{(N+2)}{6}(u_3 - \frac{1}{4}u_3^2). \tag{217b}$$

Using these results the function $\gamma^*_{\phi(3)}$ at the fixed point yields the value of η_{L4} as obtained in (207), whereas the function $\bar{\gamma}^*_{\phi^2(3)}$ at the fixed point reads

$$\bar{\gamma}^*_{\phi^2(3)} = \frac{(N+2)}{(N+8)}\epsilon_L[1 + \frac{3(N+3)}{(N+8)^2}\epsilon_L], \tag{218}$$

which is the same as that obtained in the fixed point using normalization conditions and leads to the same critical exponent ν_{L4} from (208) as the reader is invited to check. Therefore, the complete equivalence between the two renormalization schemes is assured.

Notice that the critical exponent η_{L4} for the isotropic case is different from the original result Ref. [1]. Since we have checked our results using two distinct renormalization schemes as shown above, the critical indices presented by those authors should be checked using more than one renormalization procedure in order to clarify this discrepancy.

VIII. CONCLUSIONS AND PERSPECTIVES

All the critical exponents for the m-axial Lifshitz critical behavior for the anisotropic ($1 \leq m \leq d-1$) and the isotropic ($d = m$ close to 8) cases are explicitly derived at $O(\epsilon_L^2)$. We have shown that up to the loop order considered in this work strong anisotropic scaling theory holds since the relations $\nu_{L4} = \frac{1}{2}\nu_{L2}$ and $\eta_{L4} = 2\eta_{L2}$ are exact. The exponents associated to critical correlations perpendicular to the competing axes easily reduces to the Ising-like exponents when $m = 0$, the only difference being the perturbation parameter ϵ_L replacing the usual ϵ in noncompeting systems. These relations imply that the new scaling laws, obtained here using two independent renormalization group transformations, reduce to the ones previously found by Hornreich, Luban and Shtrikman [1].

Moreover, to our knowledge this is the first time that all the exponents for the isotropic behavior are obtained explicitly through the use of the new scaling relations presented [22]. Besides, they are shown explicitly not to be recoverable from the anisotropic situation in the limit $d \to m$. The structure of the Feynman integrals in the isotropic case indicates that it deserves a especial treatment when compared with the anisotropic situation as clarified in this article.

The new results for the calculation of arbitrary loop Feynman integrals are obtained by demanding that they are homogeneous functions of arbitrary external momenta. Even though the calculations are carried out in a given order in perturbation theory, the author is convinced, however, that the conclusions hold to all orders.

The simple analytical expressions for each coefficient in the ϵ_L-expansion of the critical indices are rather encouraging to proceed the evaluation of other universal amounts, like critical amplitudes [35]. It would be interesting to compare some experimental results available for MnP like the specific heat critical amplitude ratio [17] with theoretical calculations within the context of an ϵ_L-expansion using the techniques described in the present work. In addition, a thorough RG analysis to prove that all amplitude ratios are indeed universal for the Lifshitz critical behavior was not done yet. Actually, the idea presented in this work might be suitable to demonstrate the universal character of the above mentioned critical ratios and calculate all of them.

Other problems can be pursued using the present method. The treatment of finite-size effects for the Lifshitz behavior can be devised in analogy to the noncompeting situation [39, 40]. The systems may be finite (or semi-infinite) along one (or several) of their dimensions, but they are of infinite extent in the remaining directions. Examples include systems which are finite in all directions, such as a (hyper) cube of size L, and systems which are of infinite size in $d' = d-1$ dimensions but are either of finite thickness L along the remaining direction (d-dimensional layered geometry) or of a semi-infinite extension. The presence of

geometrical restrictions on the domain of systems also requires the introduction of boundary conditions (periodic, antiperiodic, Dirichlet and Neumann) satisfied by the order parameter on the surfaces. In particular, the validity limits of the ϵ_L-expansion for these systems and the approach to bulk criticality in a layered geometry can be studied [41].

Recently, typical surface phenomena in noncompeting systems were generalized to competing systems using Monte Carlo simulations for the ANNNI model [42]. However, as far as the Lifshitz behavior is concerned, a theoretical description of these systems is still lacking. The field-theoretical framework just presented might be useful to address this problem.

The quest towards a generalization of the Lifshitz universality classes whenever arbitrary momenta powers arise in the Lagrangian (1) as the effect of further competition is quite a fascinating issue [43]. It is expected that it can be solved along the same lines described in this work [44].

In summary, we have described the Lifshitz critical behavior in its complete generality in what concerns its critical exponents. We have presented new field theory renormalization group methods which resulted in new analytical expressions for all the critical indices in the isotropic as well as in the anisotropic cases at least at $O(\epsilon_L^2)$. We hope our findings will be useful to unveil further issues related to the physics of competing systems.

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